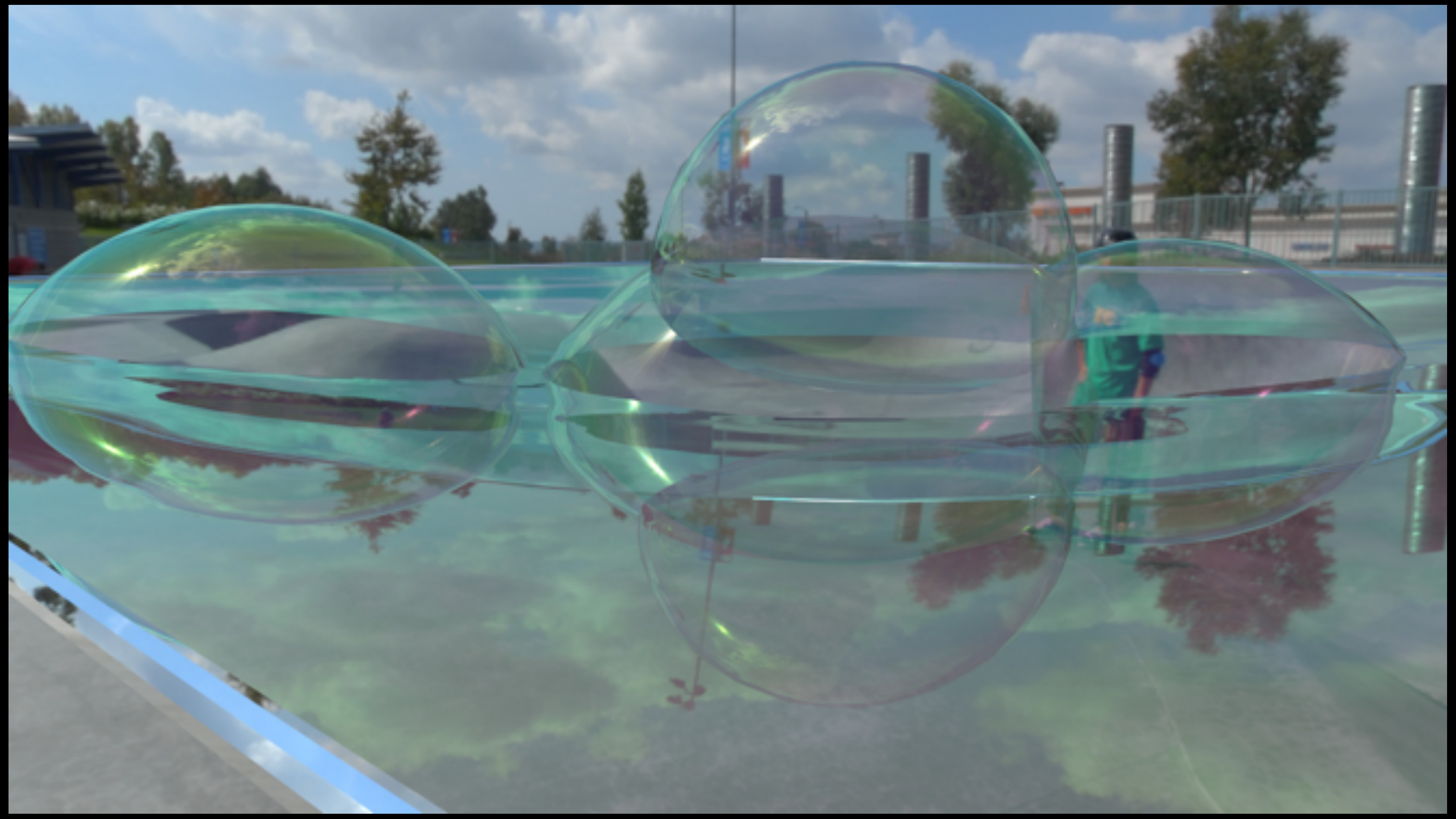
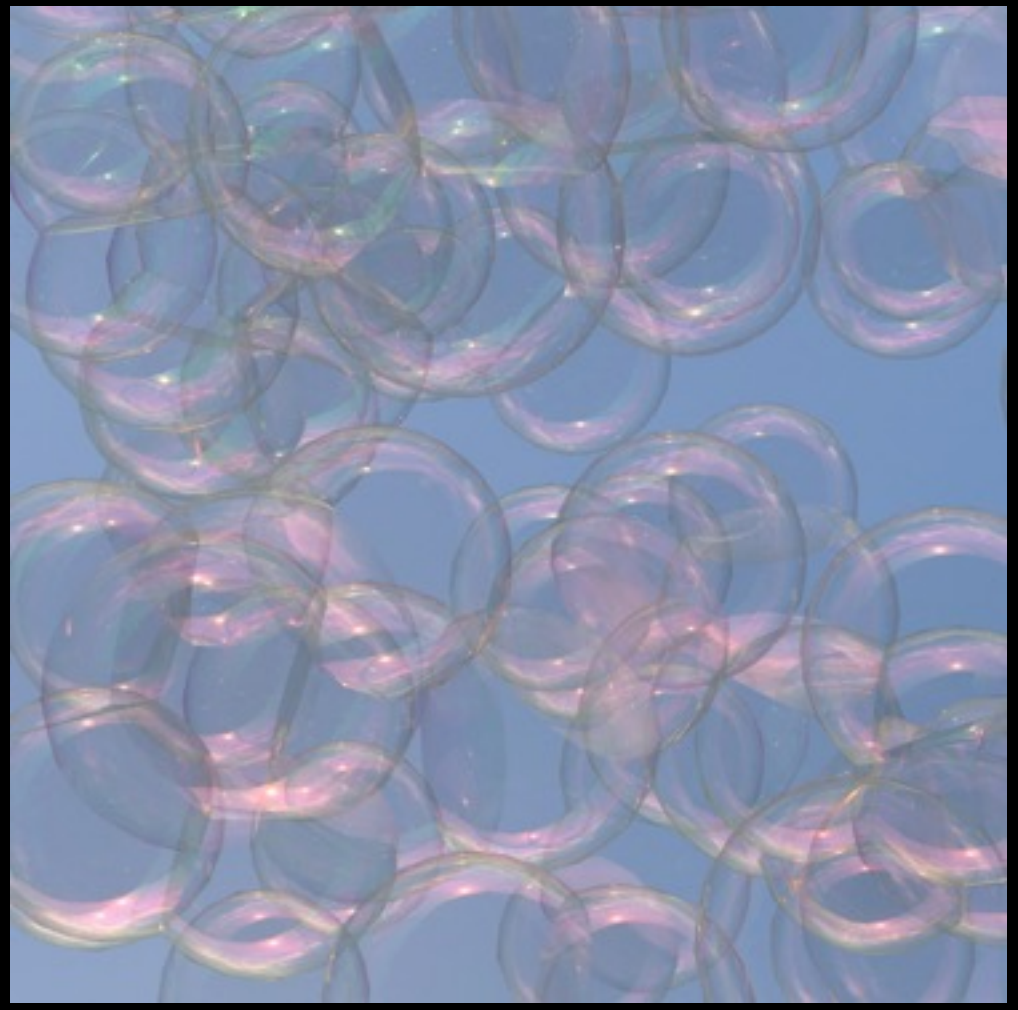
A Hyperbolic Geometric Flow for Evolving Films and Foams



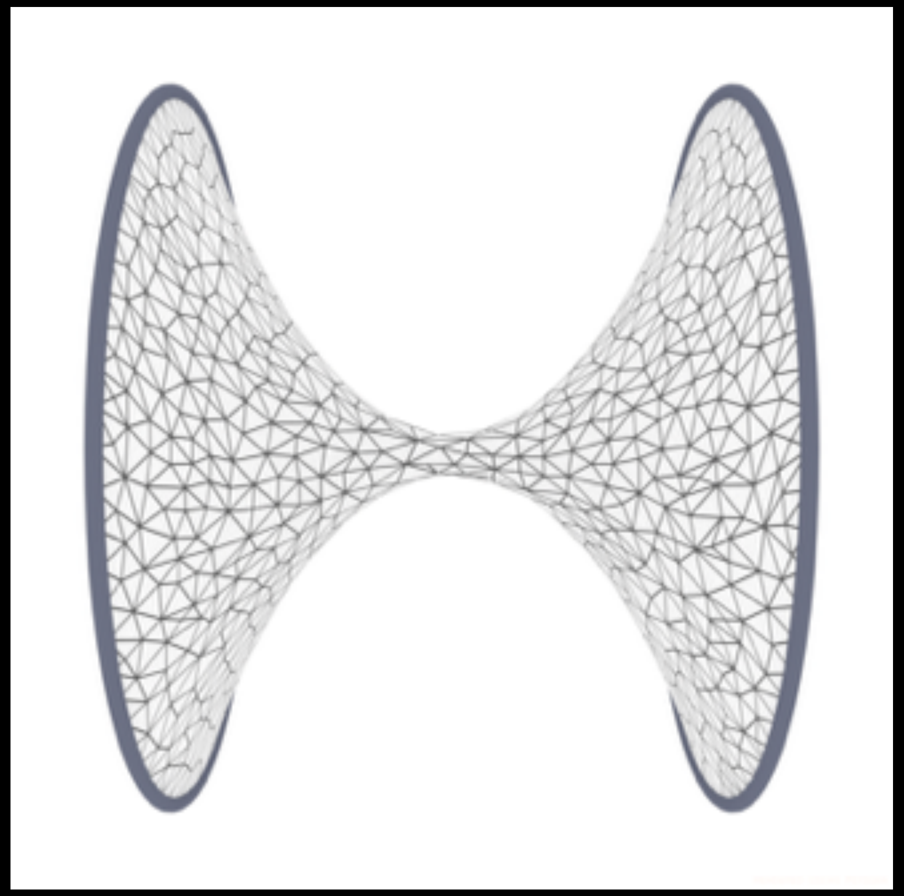


[1,2]**Sadashige Ishida**, [2]**Masafumi Yamamoto**, [3]Ryoichi Ando, and [2]Toshiya Hachisuka

1:Nikon Corporation, 2:The University of Tokyo, 3:National Institute of Informatics

Overview

- Reformulation of soap film dynamics as geometric flow
- Surface-driven simulation of soap film dynamics

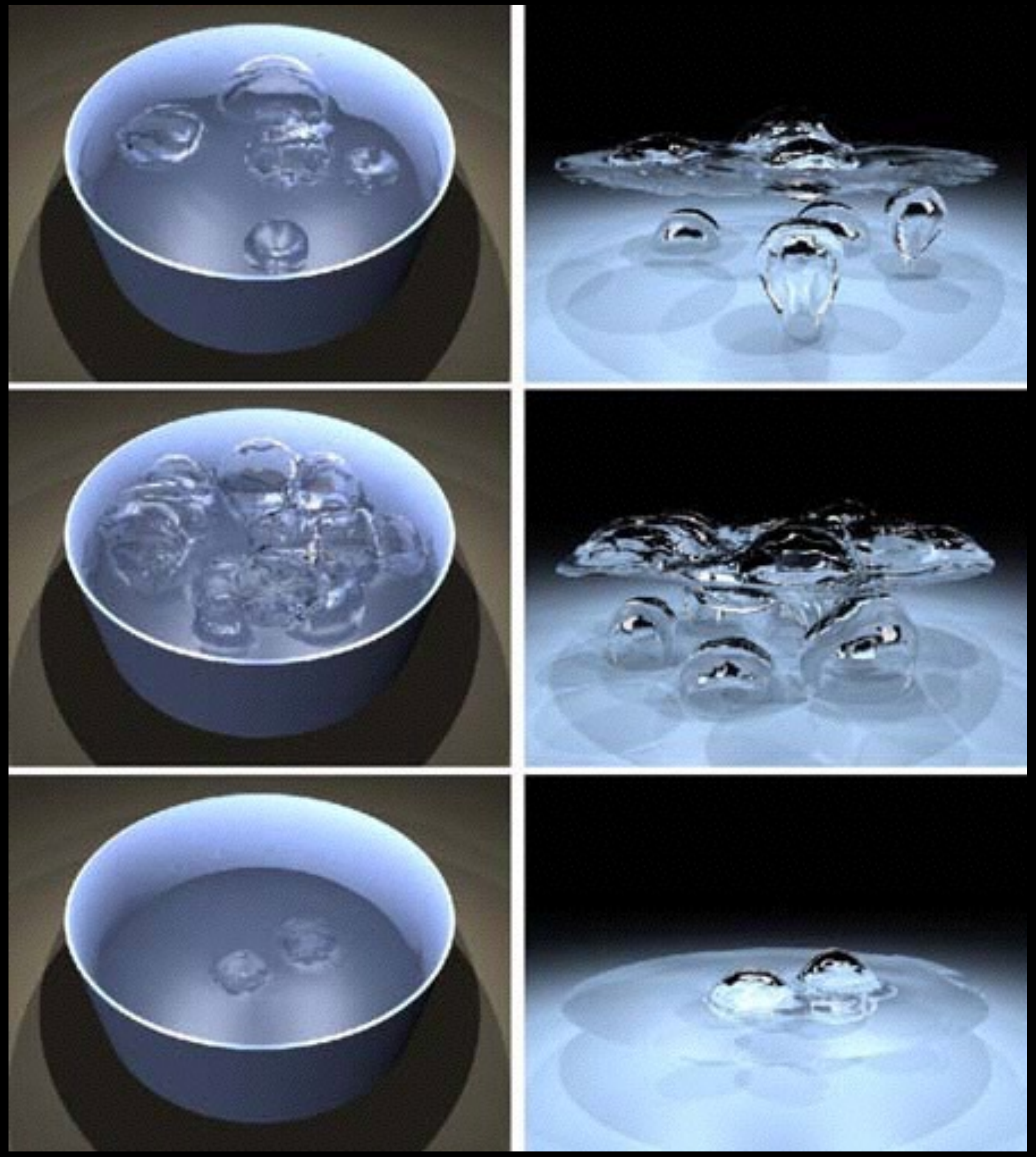


Related Work

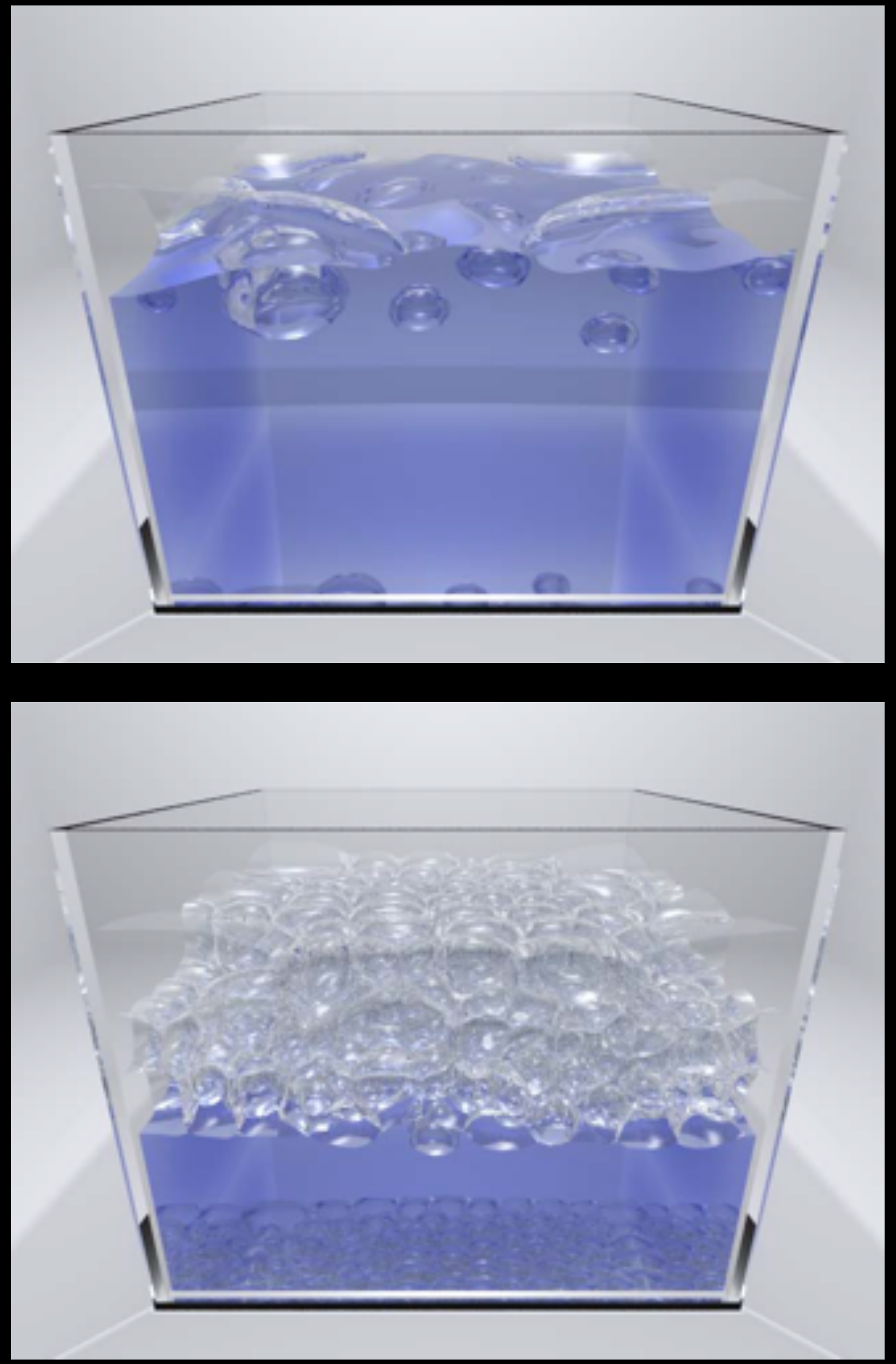
Bubble Animation

Grid-based

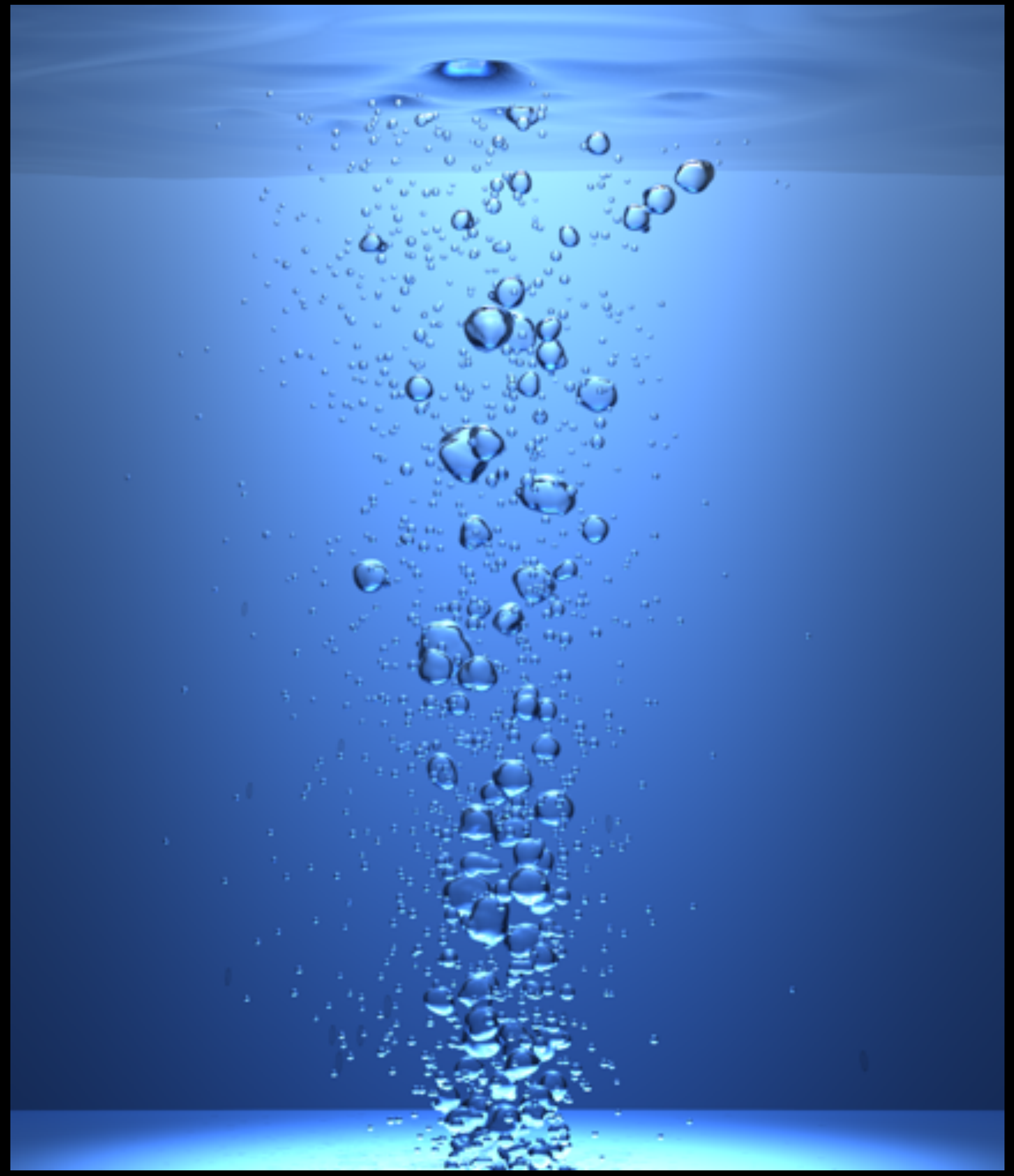
Particle-based



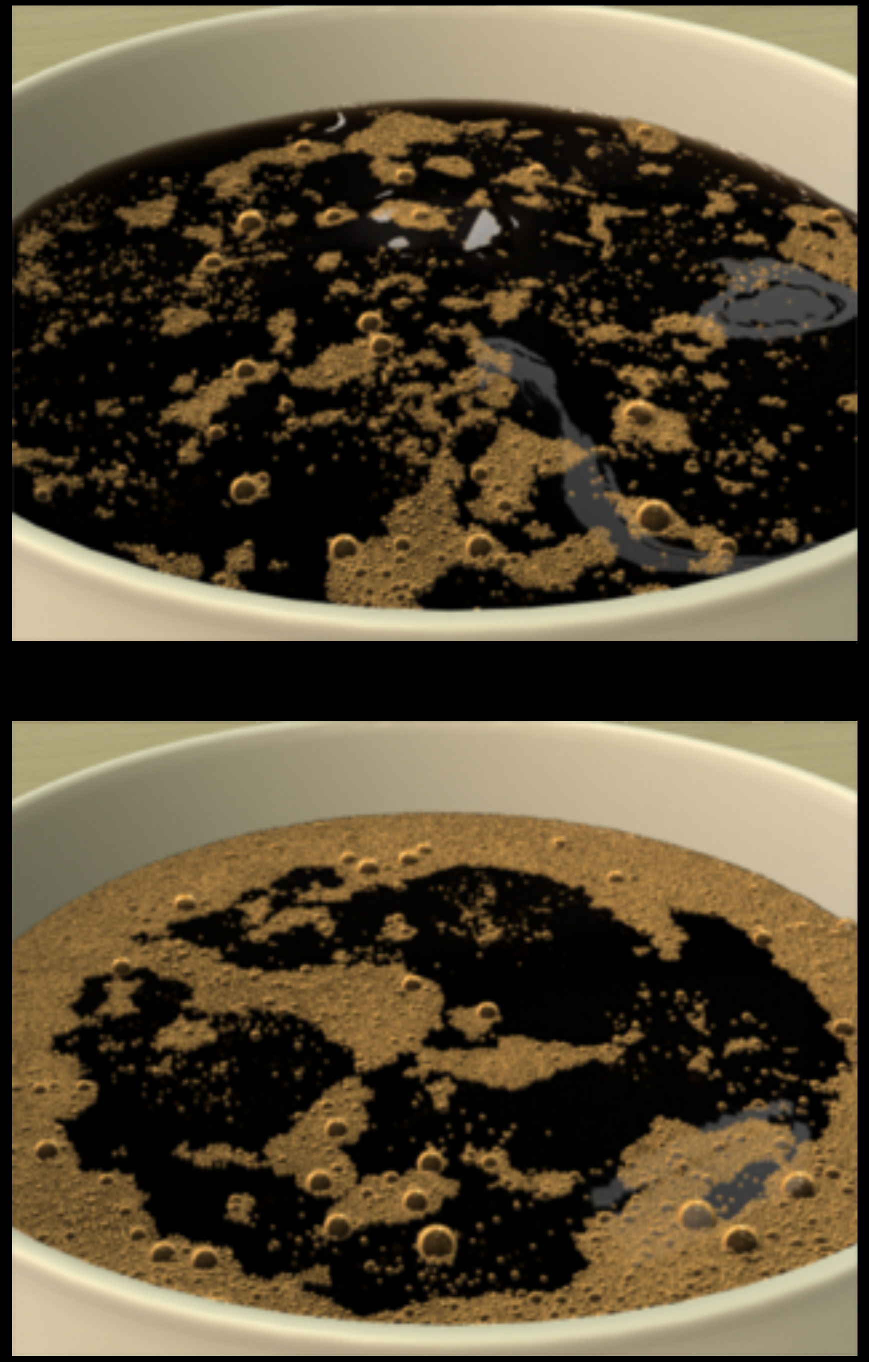
Zheng et al. [2006]



Kim et al. [2007]



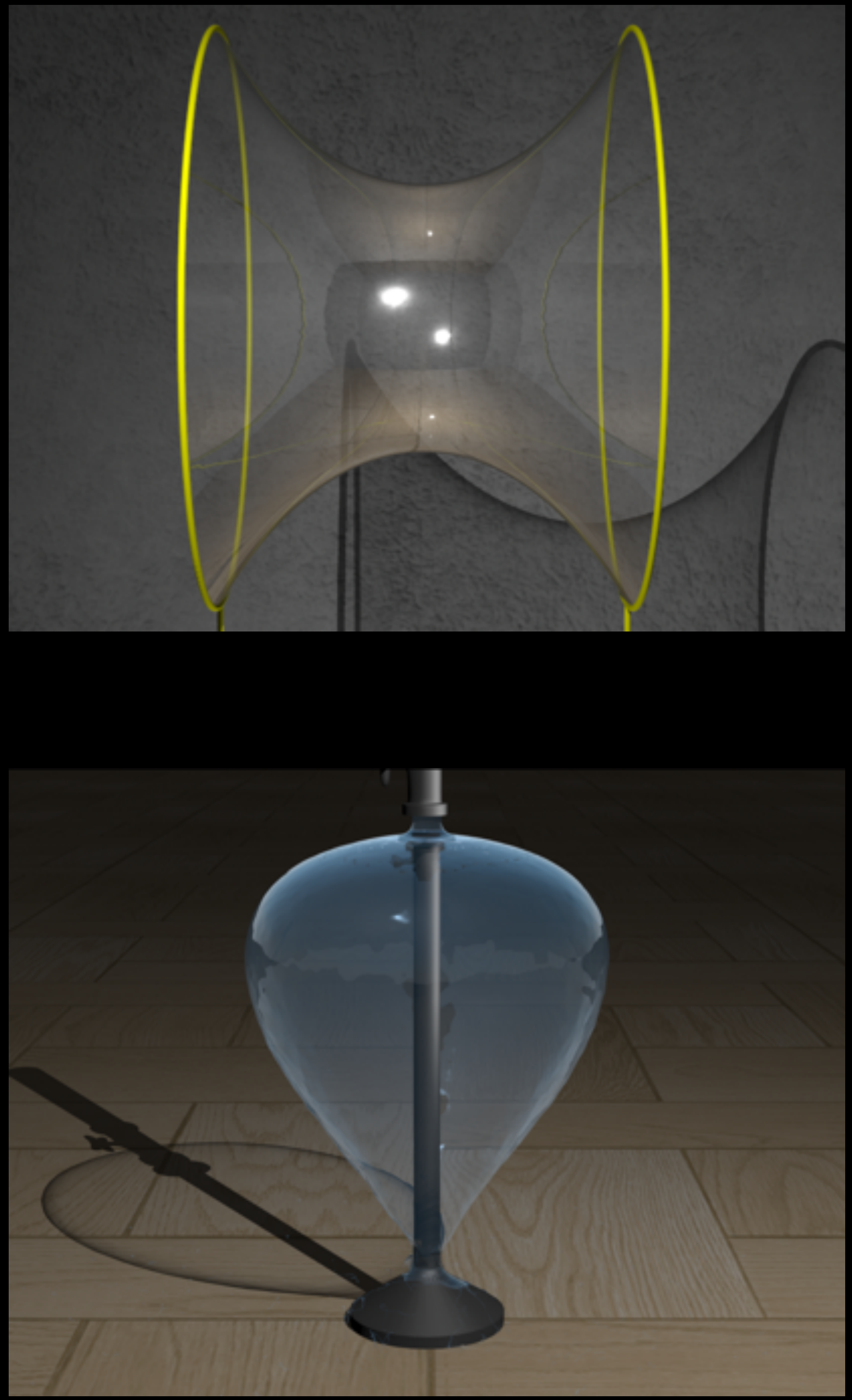
Hong et al. [2008]



Busaryev et al. [2012]

Surface-Driven Soap Films

Durikovic [2001]

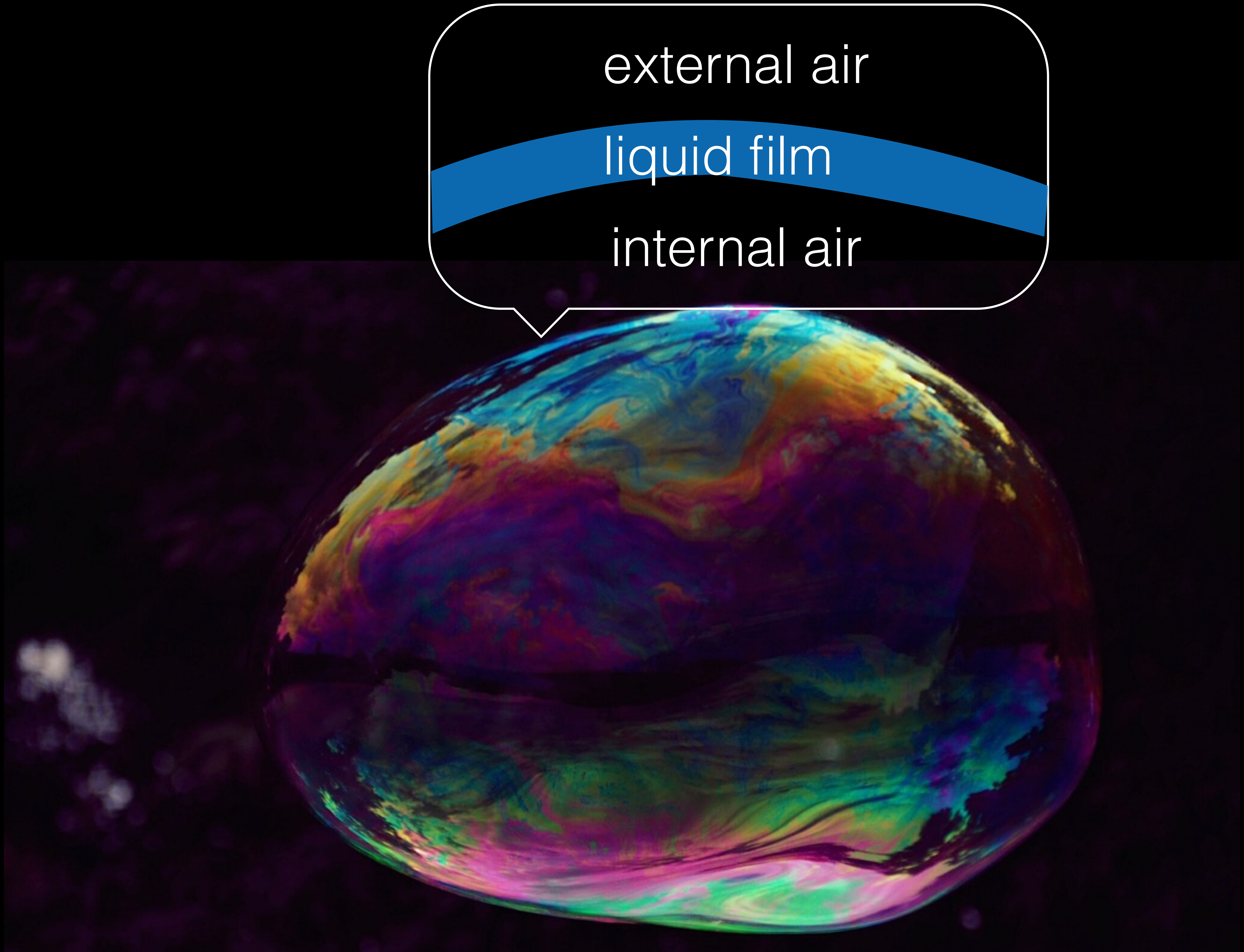


Zhu et al. [2014]

Da et al. [2015]

Soap Films in Physics

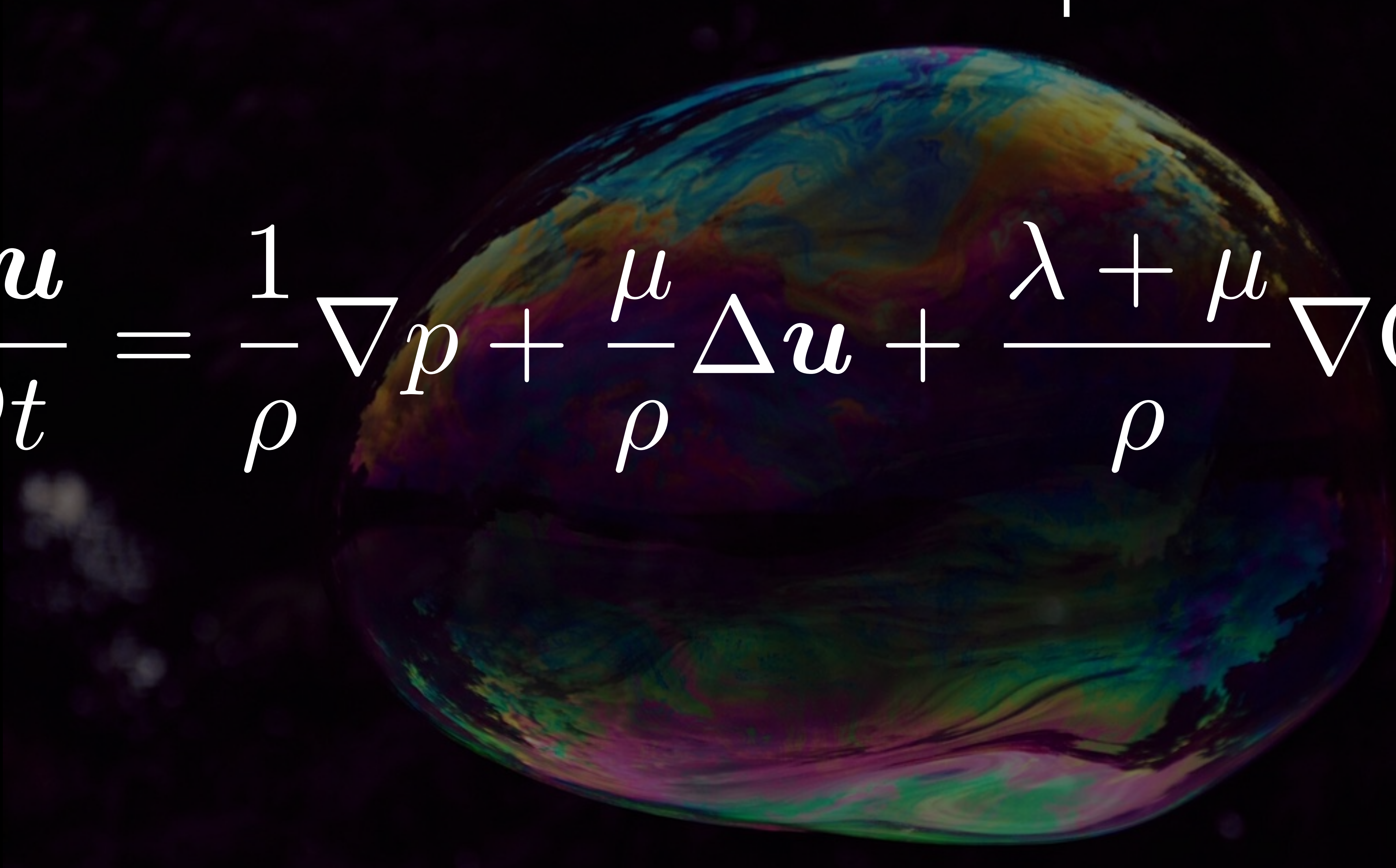
Dynamic Fluids with Three Layers



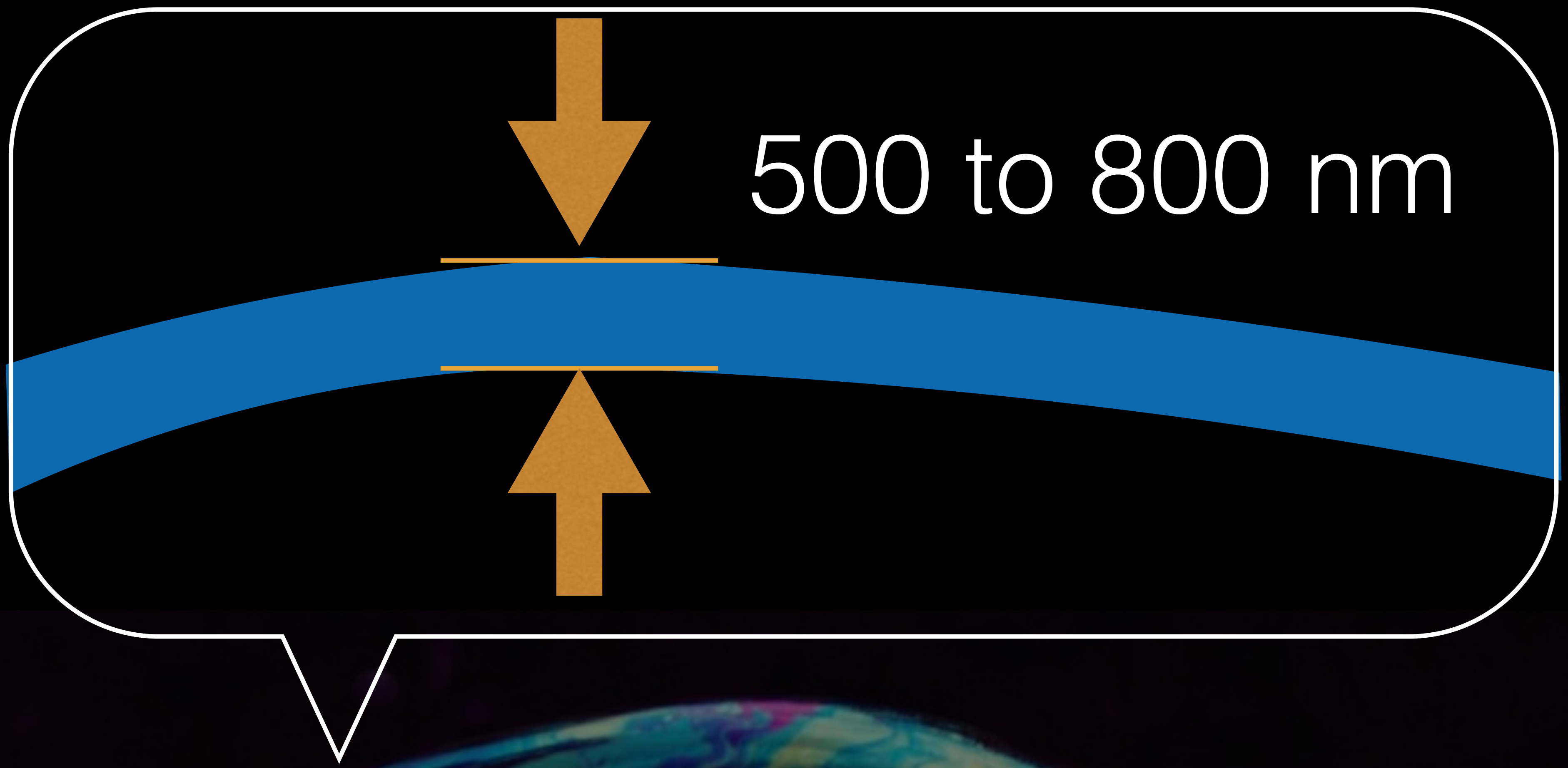
Simulation

Ideally, computable via
the Navier-Stokes equations

$$\frac{D\boldsymbol{u}}{Dt} = \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\Delta \boldsymbol{u} + \frac{\lambda + \mu}{\rho}\nabla \Theta + \boldsymbol{g}$$



Challenges



Thickness of films is extremely thin

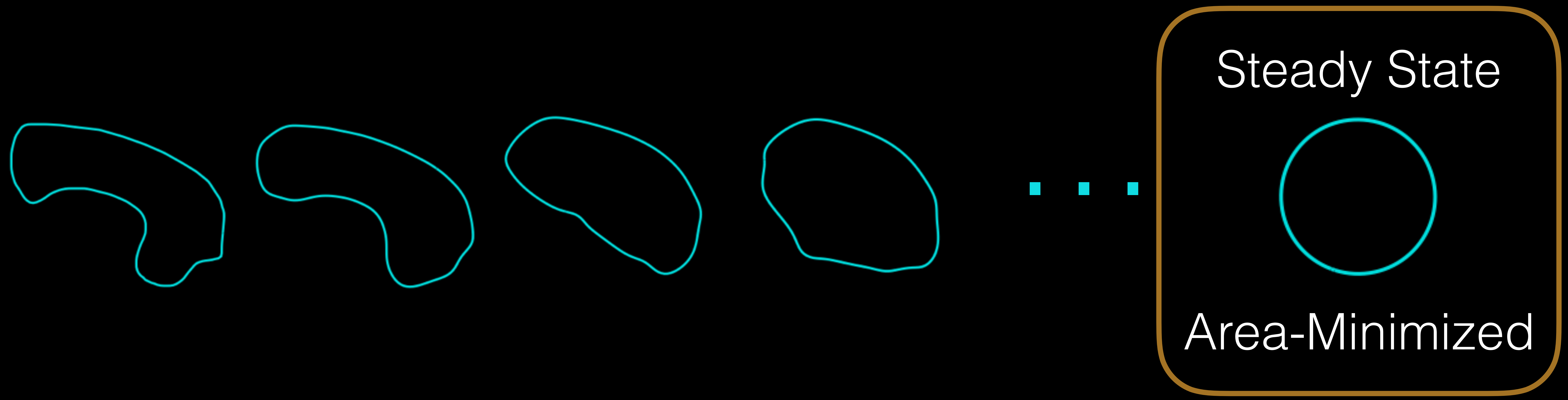
➡ Super-high resolution grid is necessary

➡ Volumetric computation is too expensive

Soap Films in Mathematics

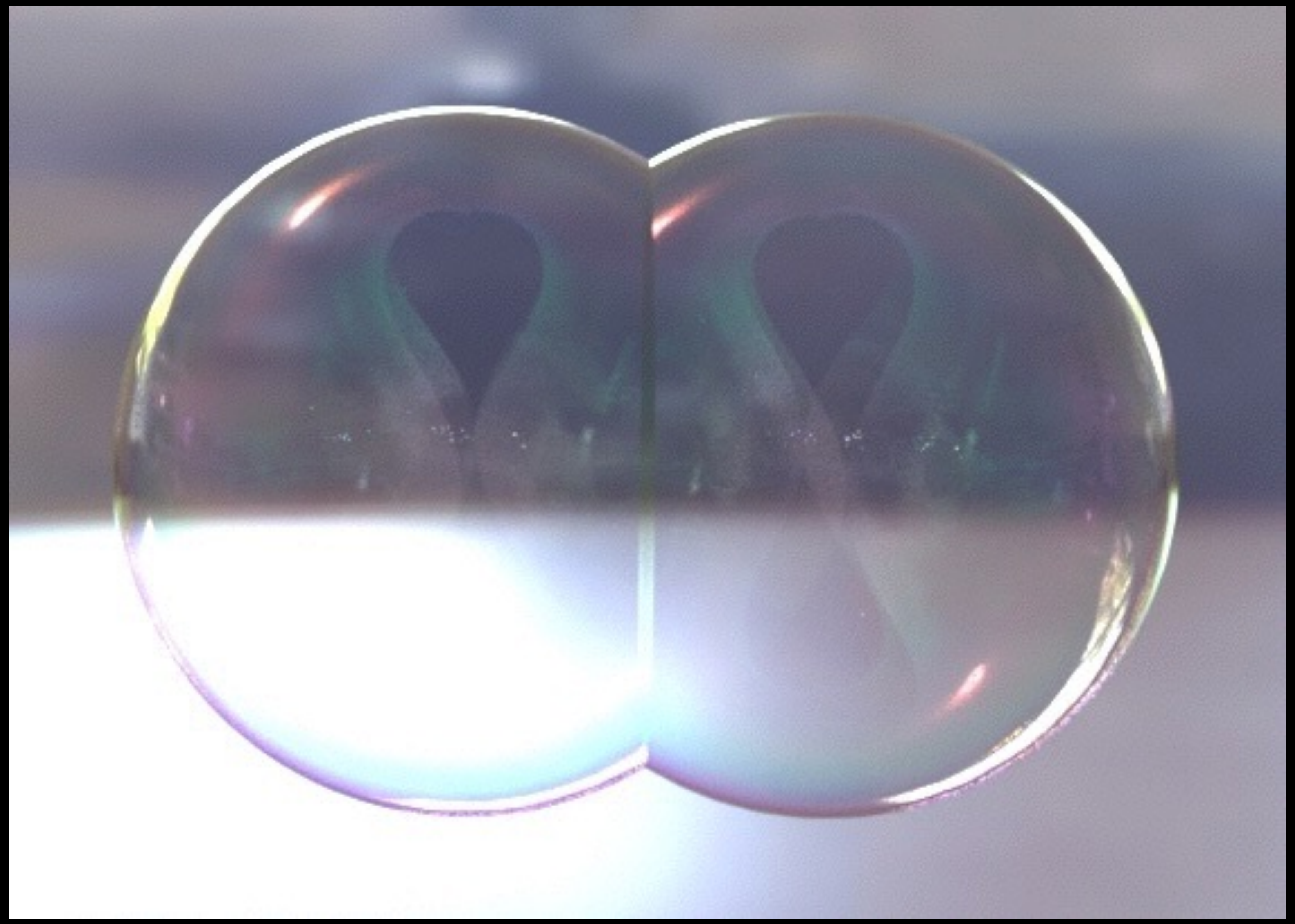
Geometric Property

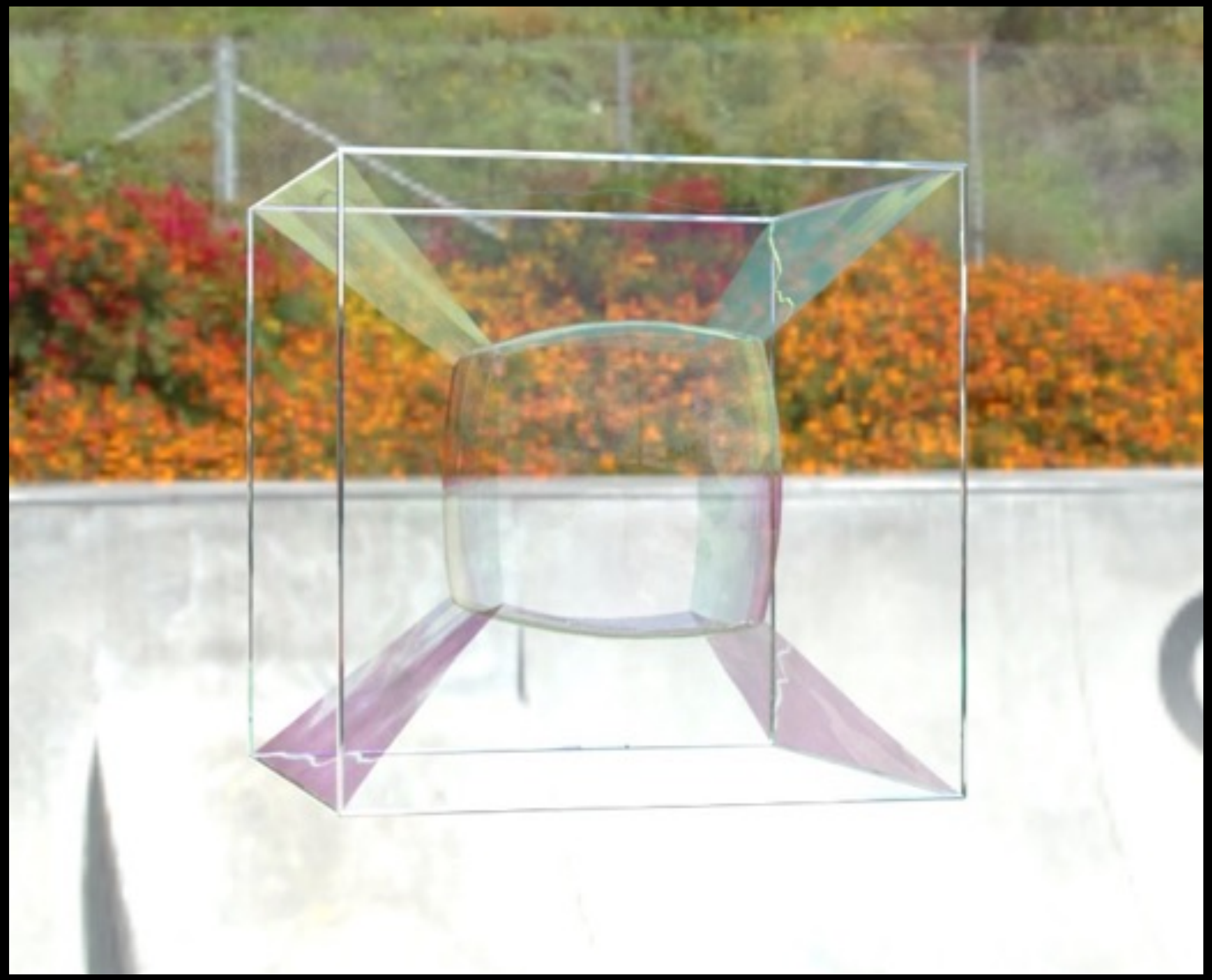
Soap films evolve to reduce their surface area, while preserving inner volumes.



Steady States and Plateau's Laws

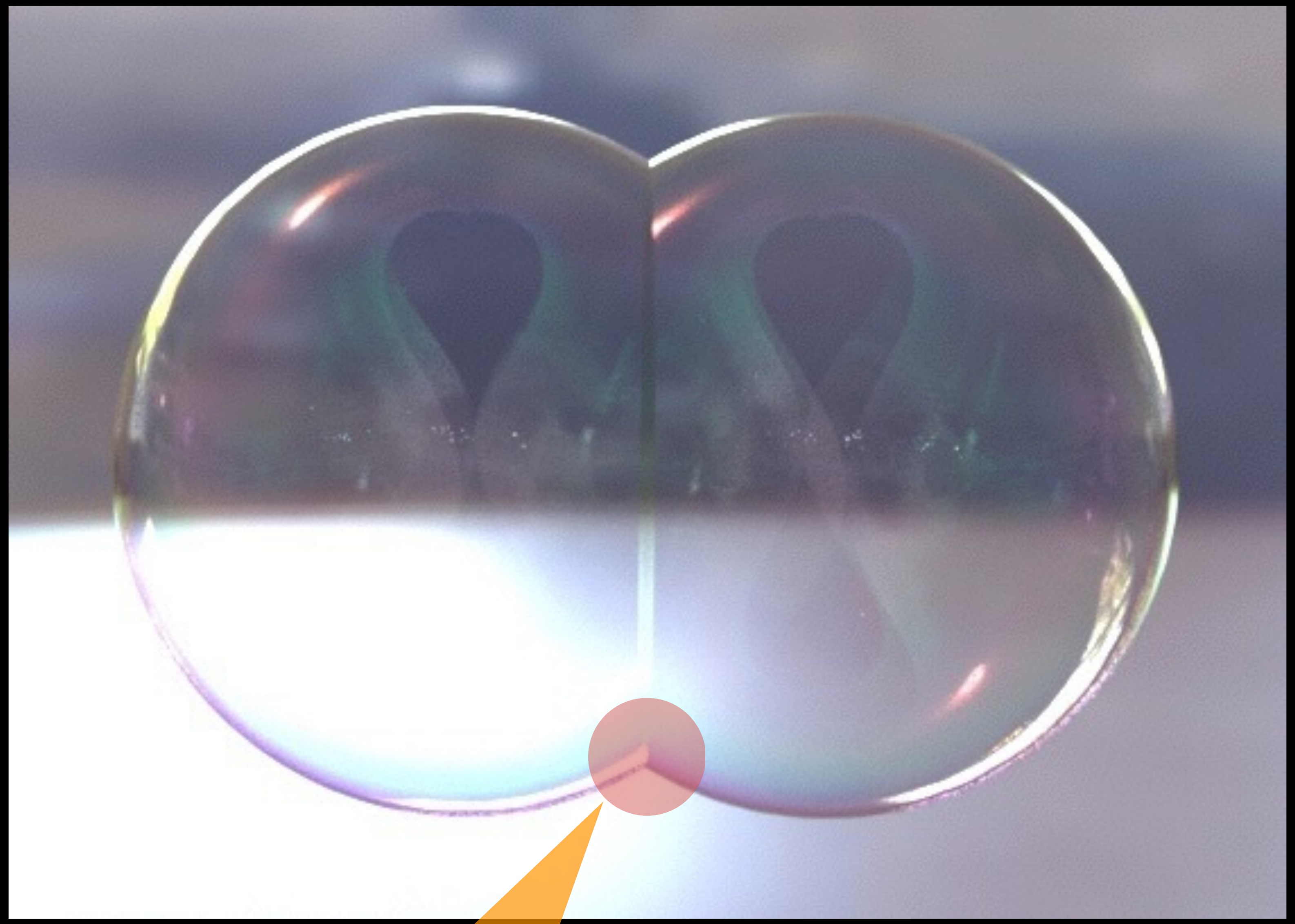
Area-minimized shapes with volume constraints



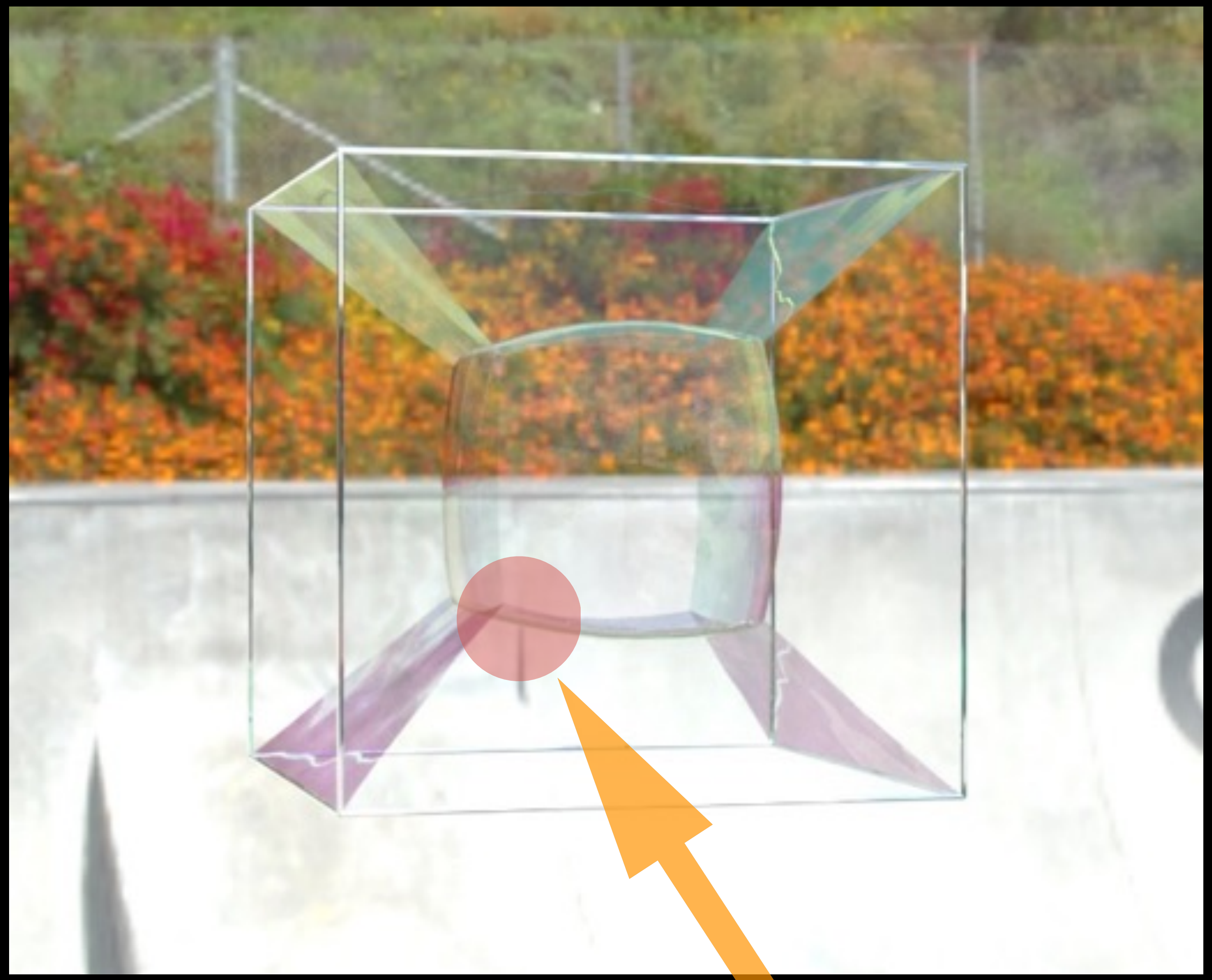


Steady States and Plateau's Laws

Area-minimized shapes with volume constraints



$\arccos(-1/2) = 120^\circ$



$\arccos(-1/3) \approx 109^\circ$

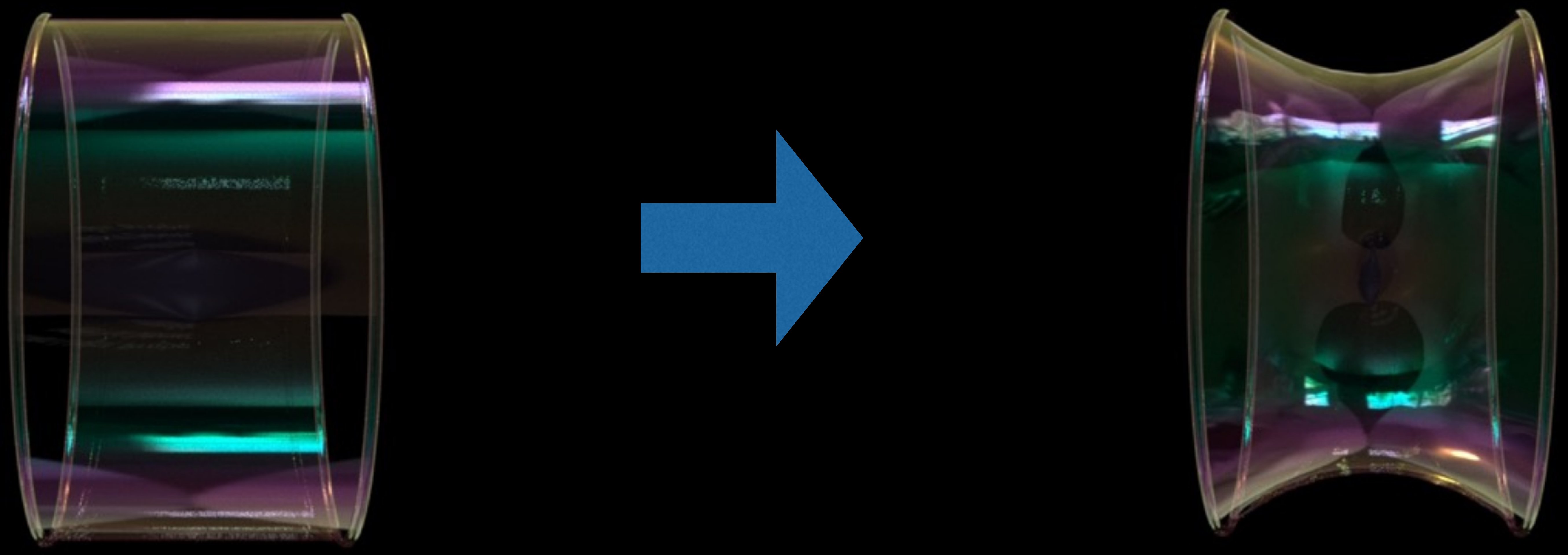
Plateau's Problem

Mathematical formulation of
the steady states

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_U |S_u^\epsilon \times S_v^\epsilon| du dv = 0$$

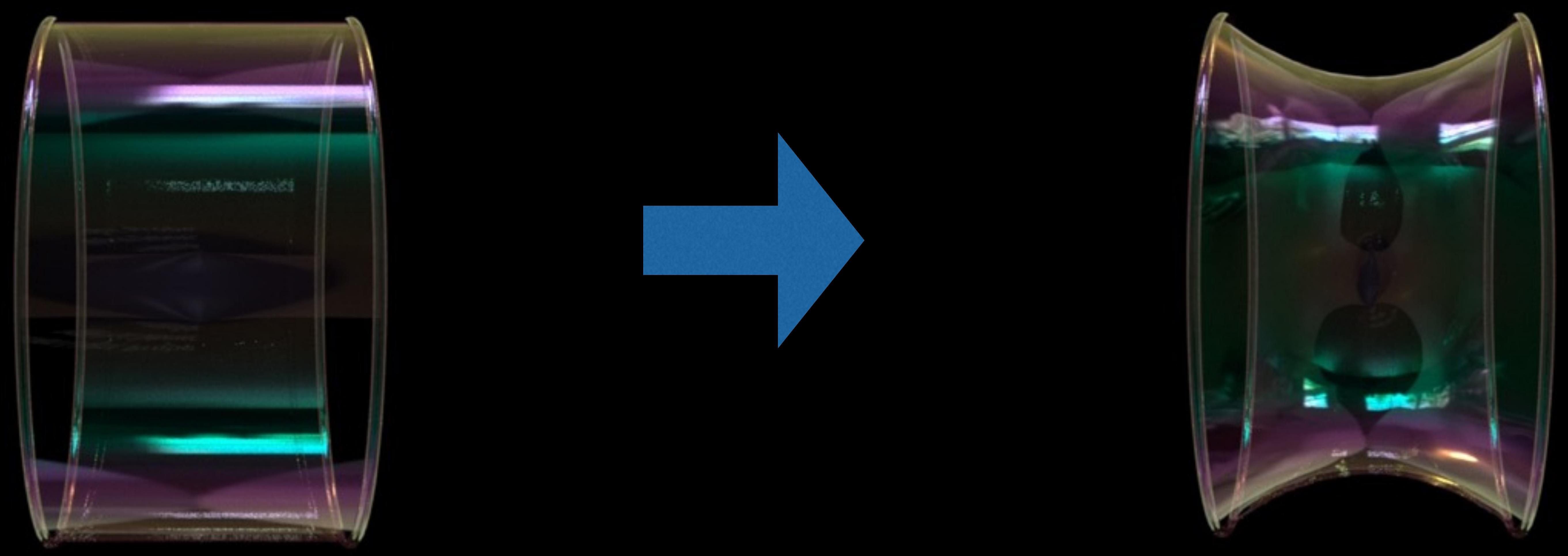
General solutions are not found yet.

Solving Plateau's Problem



Solving Plateau’s Problem

A common approach:
Evolve surface under Mean Curvature Flow.



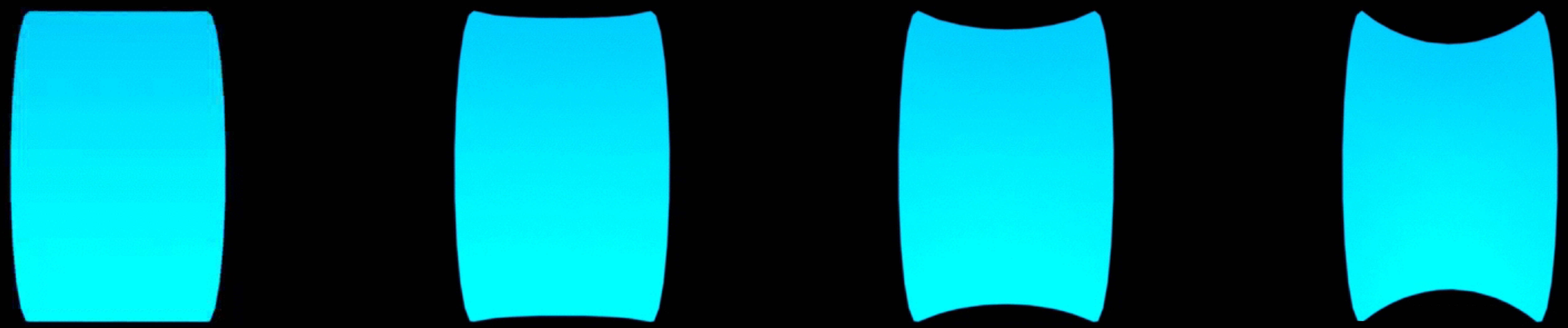
Mean Curvature Flow (MCF)

$$\frac{d\boldsymbol{x}}{dt} = -H(\boldsymbol{x},t)\boldsymbol{n}(\boldsymbol{x},t)$$

mean curvature

surface normal

Property of MCF



Single open surface ➔ Local minimum of the area functional

Property of MCF

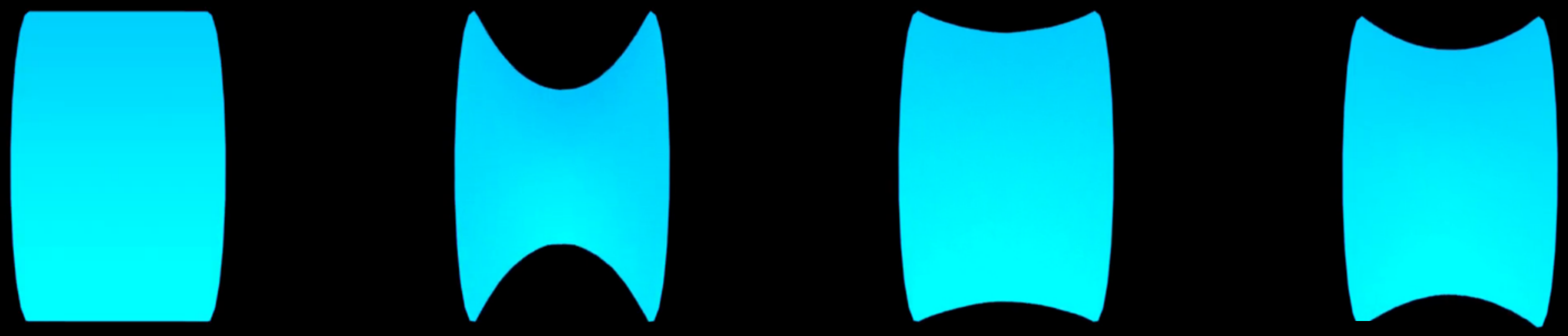
- leads films to the steady states
- no oscillation

Hyperbolic Mean Curvature Flow (HMCF)

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\beta H(\boldsymbol{x}, t)\boldsymbol{n}(\boldsymbol{x}, t)$$

constant mean curvature surface normal

Property of HMCF



Single open surface → Local minimum of the area functional

We integrate this geometric view
into the film dynamics.

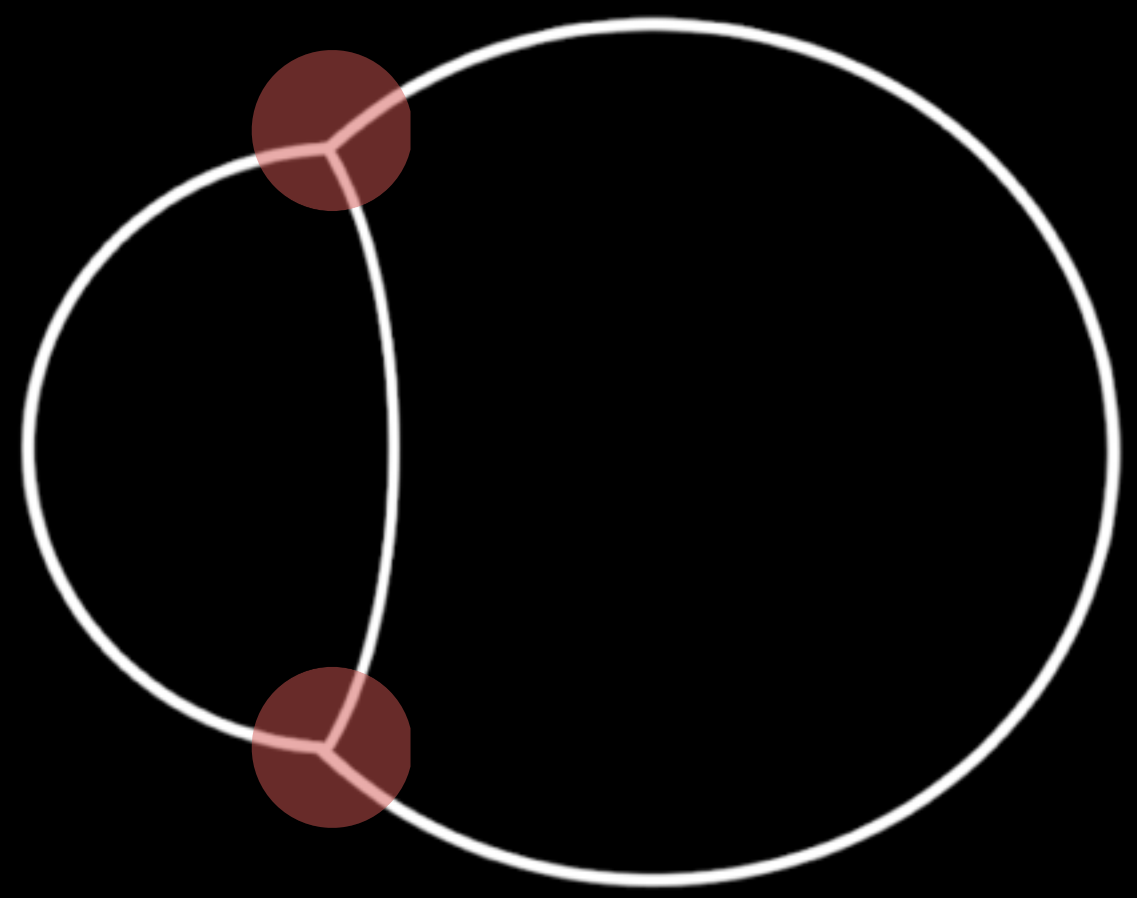
HMCF seems very good for soap film dynamics.

- leads films to the steady states
- with oscillation

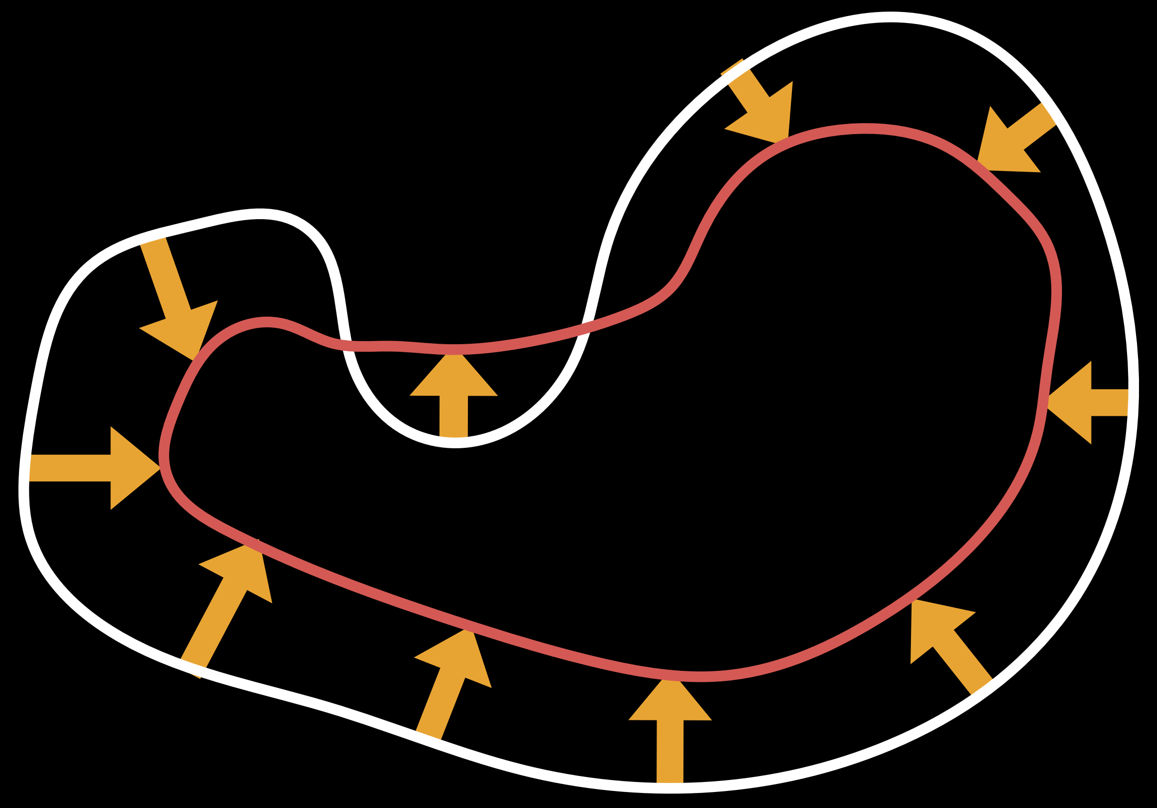
HMCF seems very good for soap film dynamics, but …

Issues

- Mean curvature is undefined on non-manifold junctions.
- HMCF does not preserve inner volume.



Non-manifold junctions



Evolution under HMCF

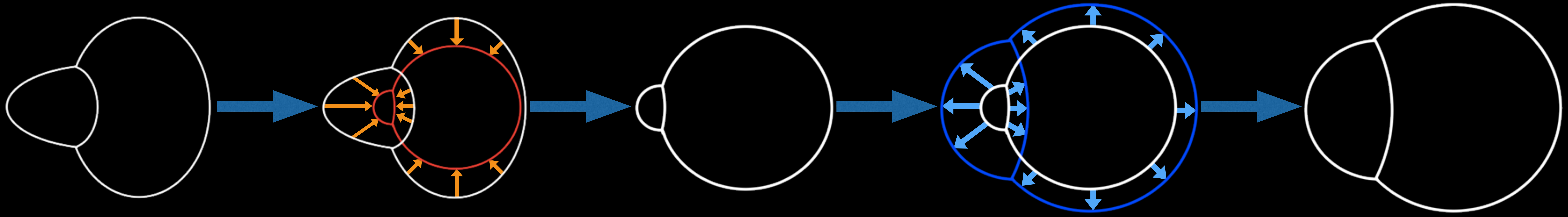
Our Solution

- Use variational derivative of the area functional, instead of mean curvature normal

$$H\boldsymbol{n} \longrightarrow \frac{\partial \mathcal{A}}{\partial \boldsymbol{x}}$$

- Volume preservation for multiple regions

Model Overview



Initial state

Intermediate state

Next state

Evolution by $\frac{\partial A}{\partial x}$

Volume preservation

Hyperbolic Mean Curvature Flow

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\beta H(x,t)n(x,t)$$

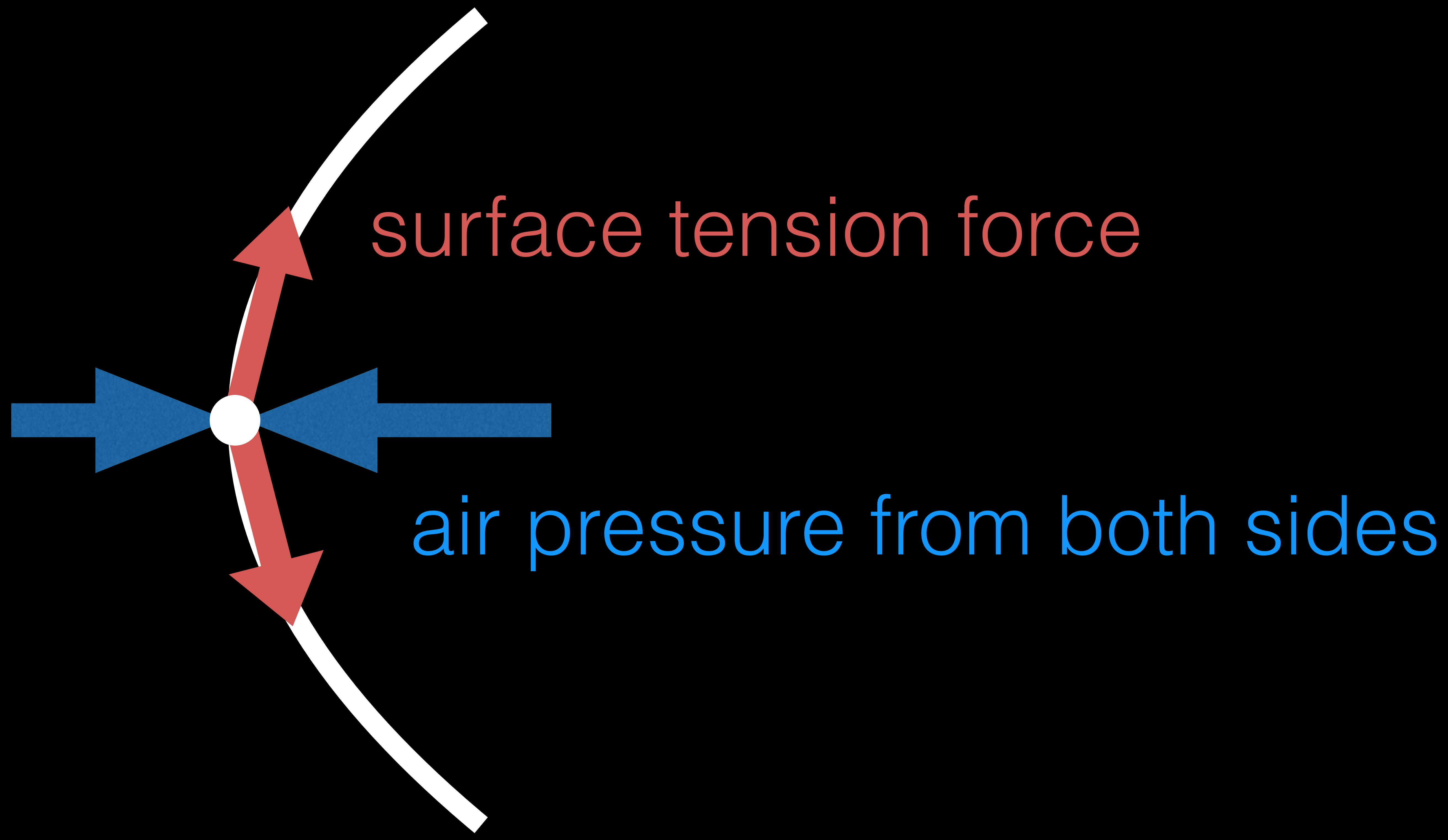
Volume Preserving Hyperbolic Geometric Flow for Multiple Surfaces

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\beta\frac{\partial A}{\partial x} + \Delta p n$$

pressure difference across films

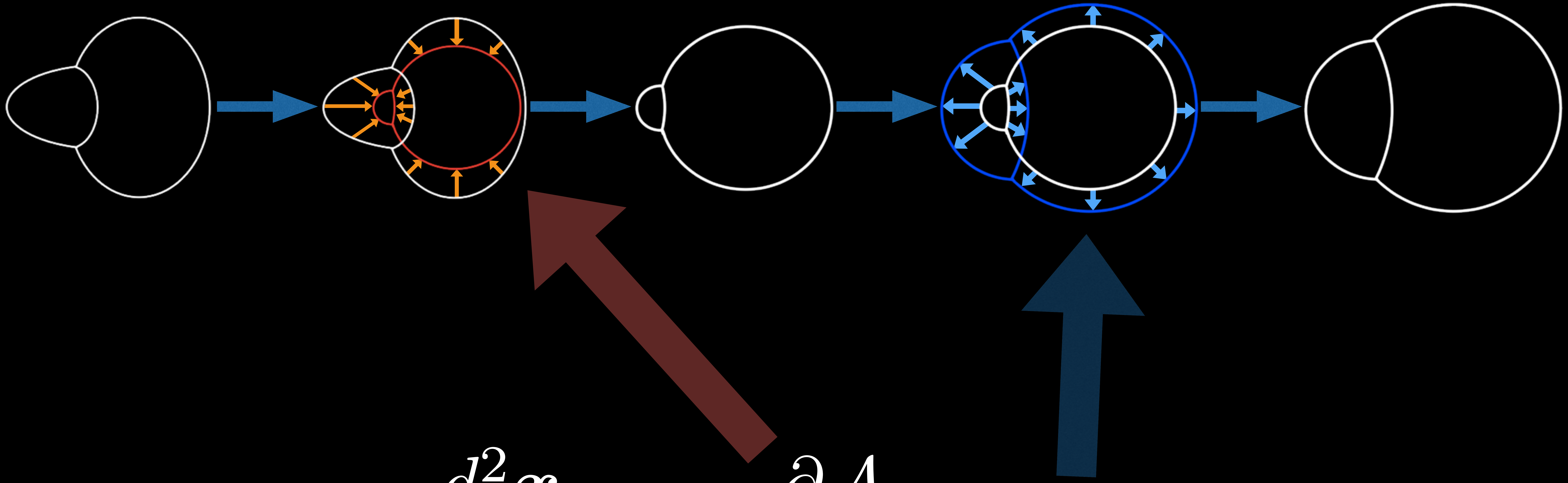
Volume Preserving Hyperbolic Geometric Flow for Multiple Surfaces

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\beta\frac{\partial\mathcal{A}}{\partial\boldsymbol{x}} + \Delta p\boldsymbol{n}$$



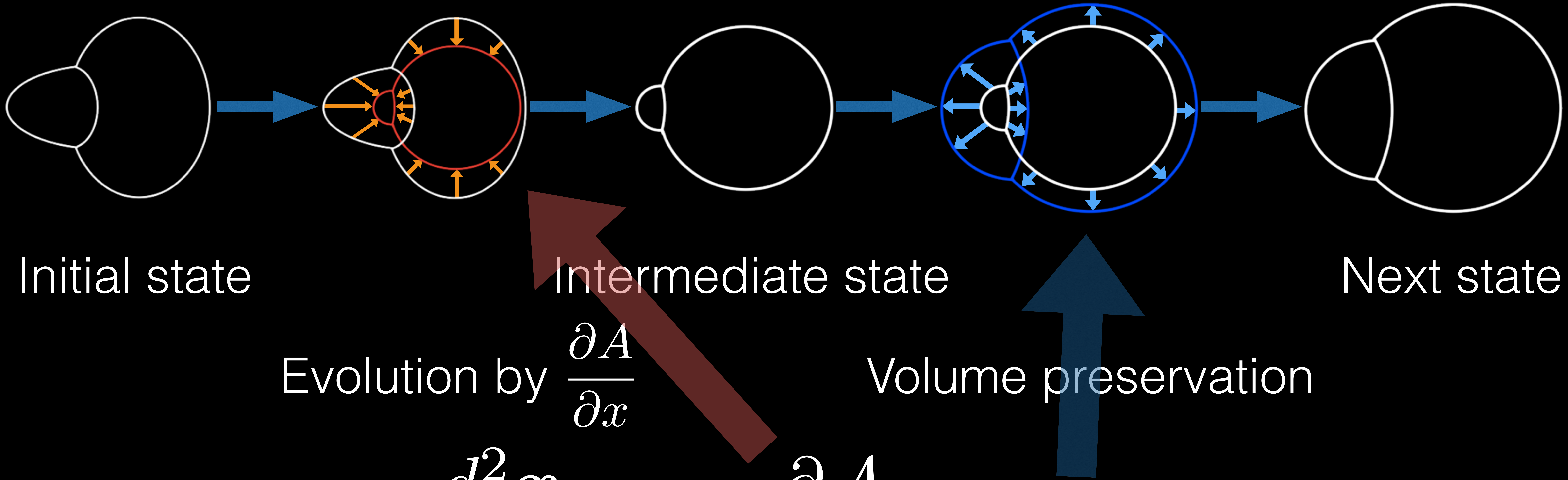
The Algorithm

Algorithm Overview



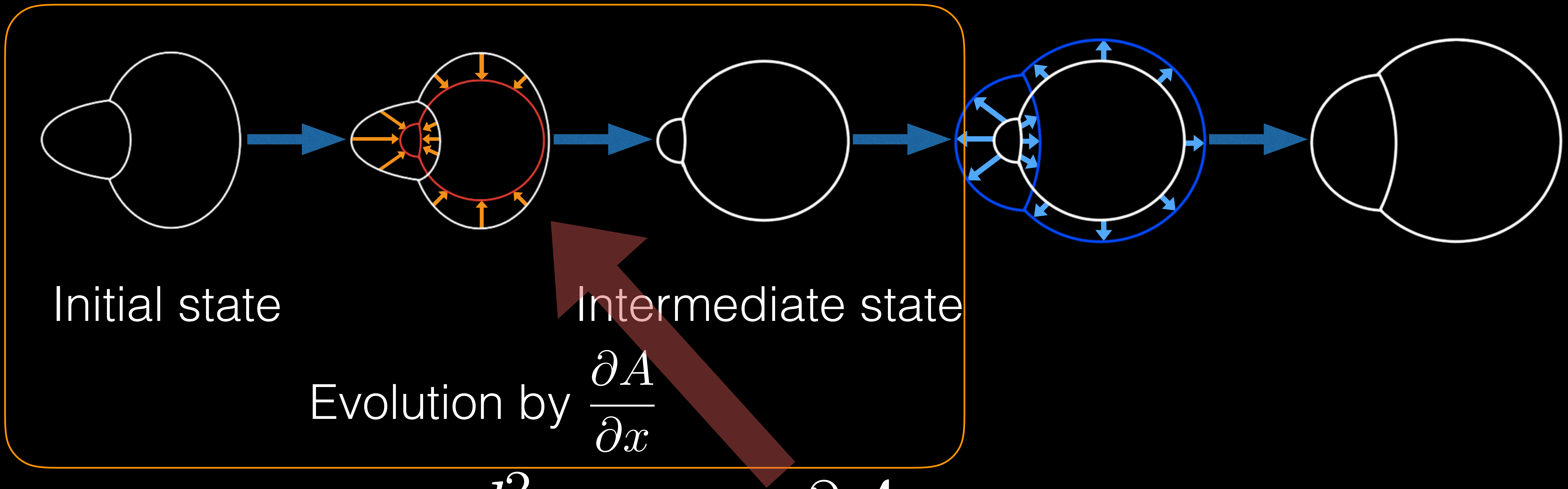
$$\frac{d^2 \boldsymbol{x}}{dt^2} = -\beta \frac{\partial A}{\partial \boldsymbol{x}} + \Delta p \boldsymbol{n}$$

Algorithm Overview



$$\frac{d^2\boldsymbol{x}}{dt^2} = -\beta\frac{\partial A}{\partial \boldsymbol{x}} + \Delta p\boldsymbol{n}$$

Evolution by $\frac{\partial A}{\partial x}$



$$\frac{d^2 \boldsymbol{x}}{dt^2} = -\beta \frac{\partial A}{\partial \boldsymbol{x}} + \Delta p \boldsymbol{n}$$

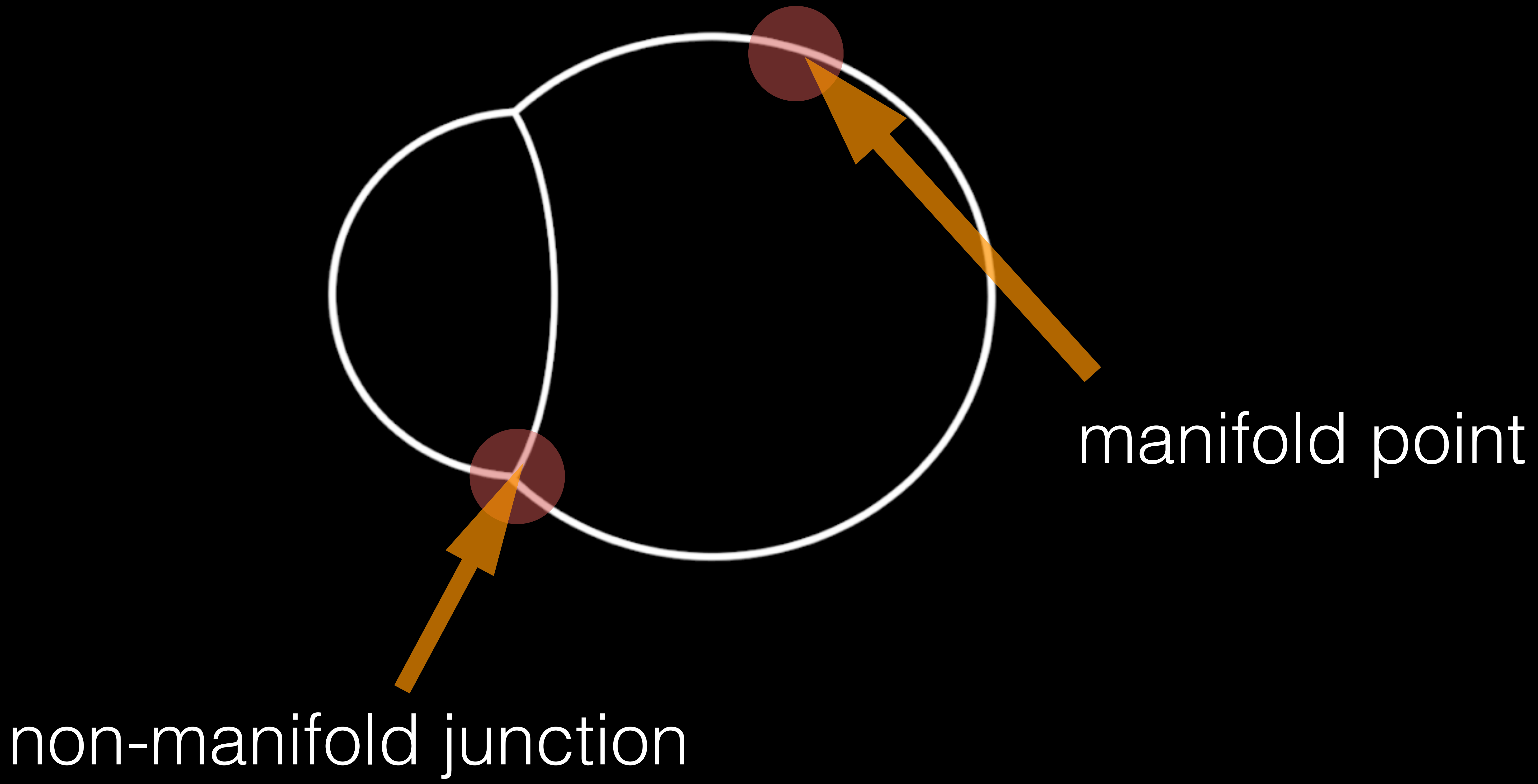
Variational Derivative of Area

$$\frac{\partial \mathcal{A}}{\partial \boldsymbol{x}}$$

Gradient of the surface area on each point

Moving $\boldsymbol{x}$ toward $-\frac{\partial \mathcal{A}}{\partial x}$ minimizes the area.

Defined Everywhere on Films



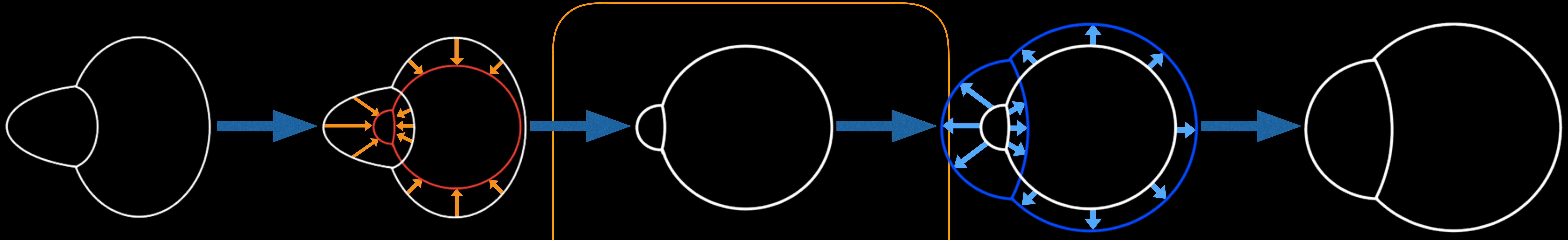
Natural Extension of $H\boldsymbol{n}$

Common properties of $\frac{\partial \mathcal{A}}{\partial \boldsymbol{x}}$ and $H\boldsymbol{n}$

- Negative direction: minimizes the local area
- Magnitude: difference from the area-minimized configuration

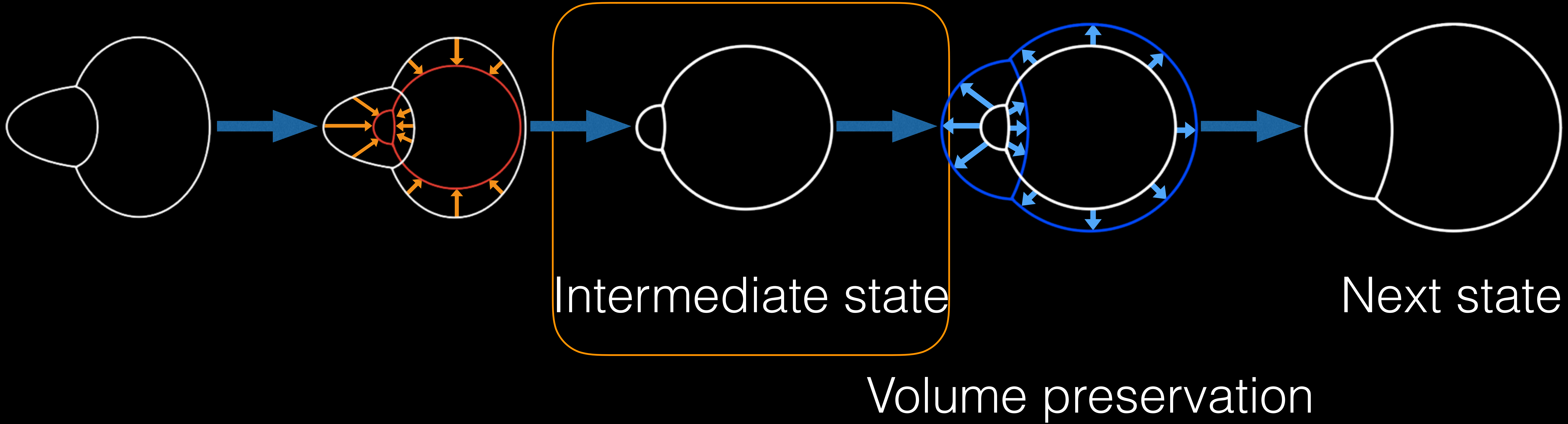
Indeed, $\frac{\partial \mathcal{A}}{\partial \boldsymbol{x}} = H\boldsymbol{n}$ on manifold points

After the First Step



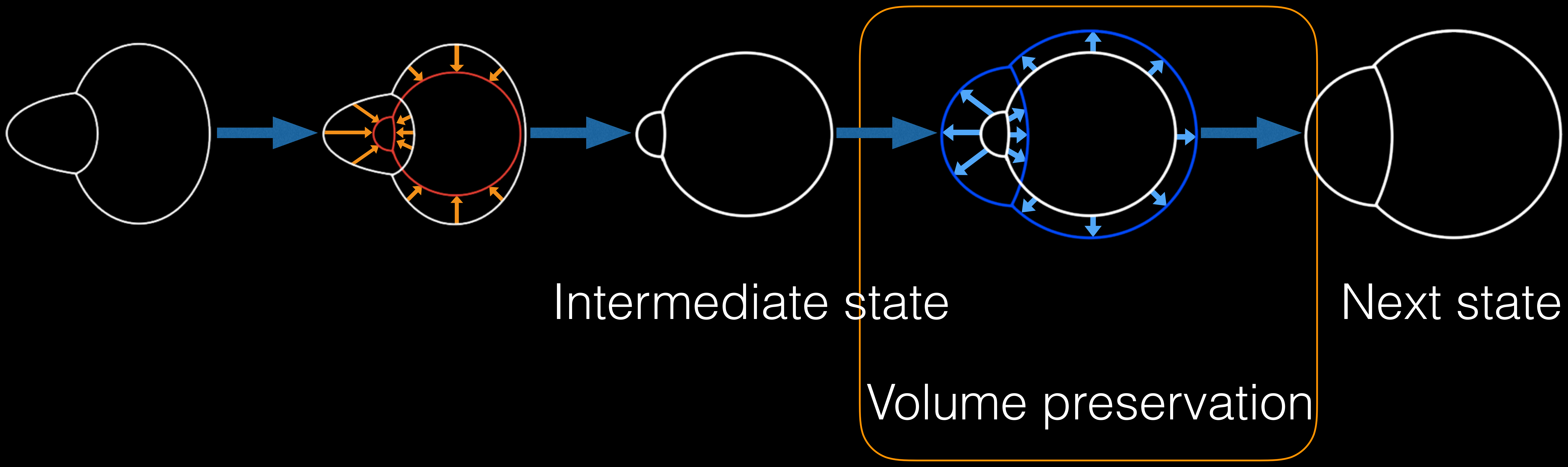
Intermediate state

After the First Step



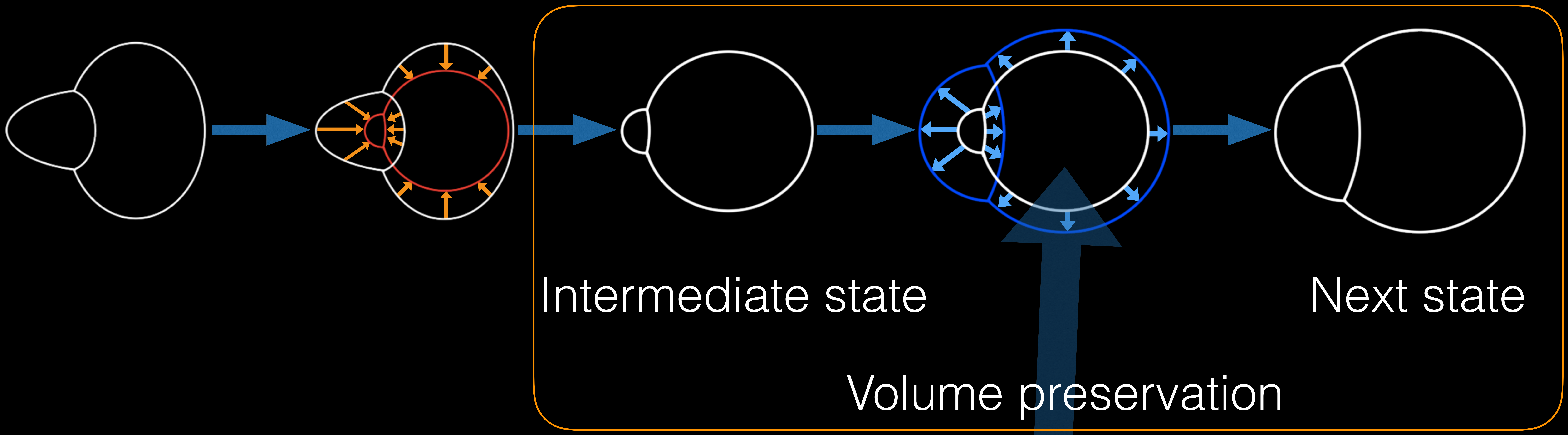
Enclosed volumes may decrease at this point.

After the First Step



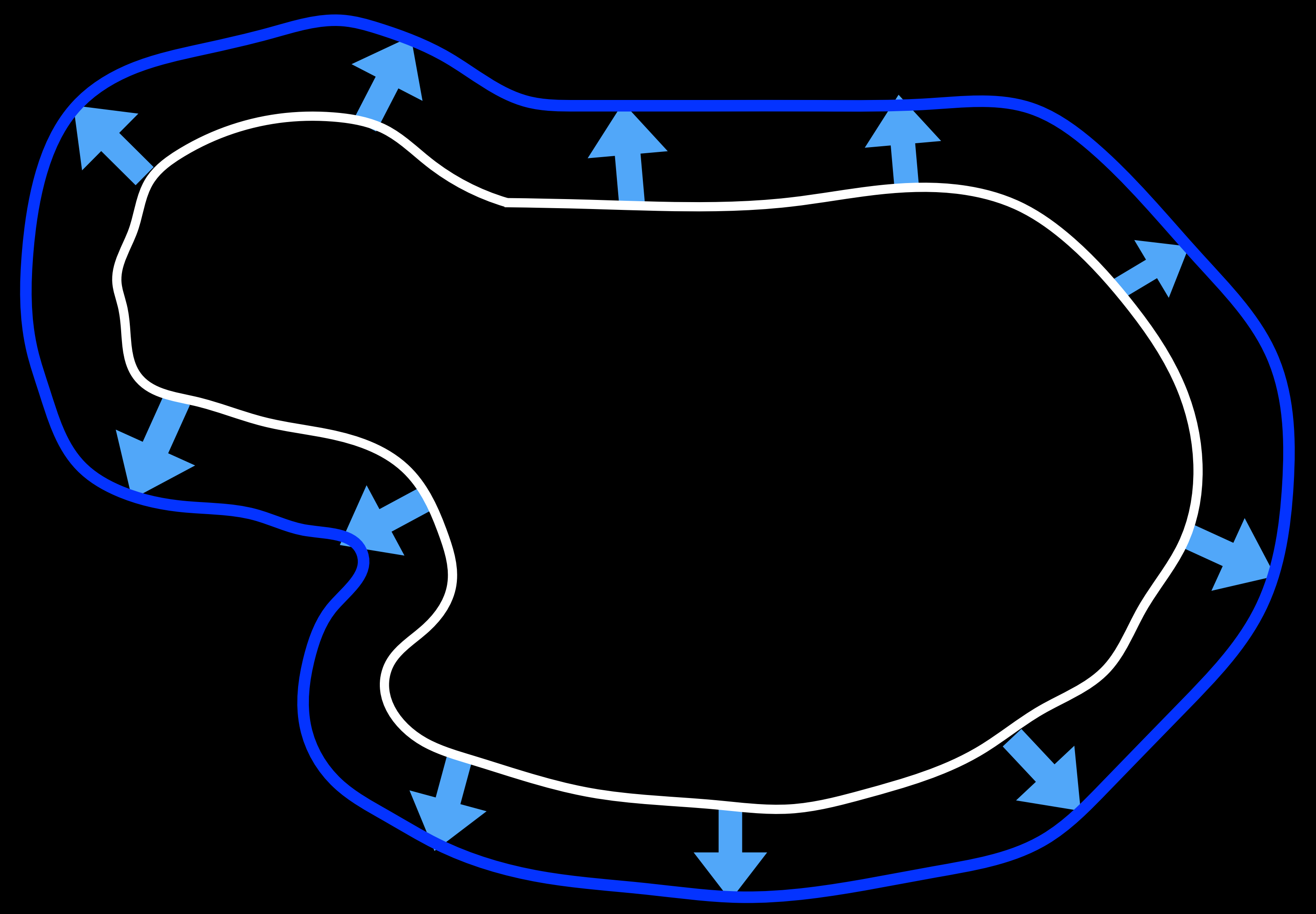
Need to resolve the volume change for all the regions.

Volume Preservation



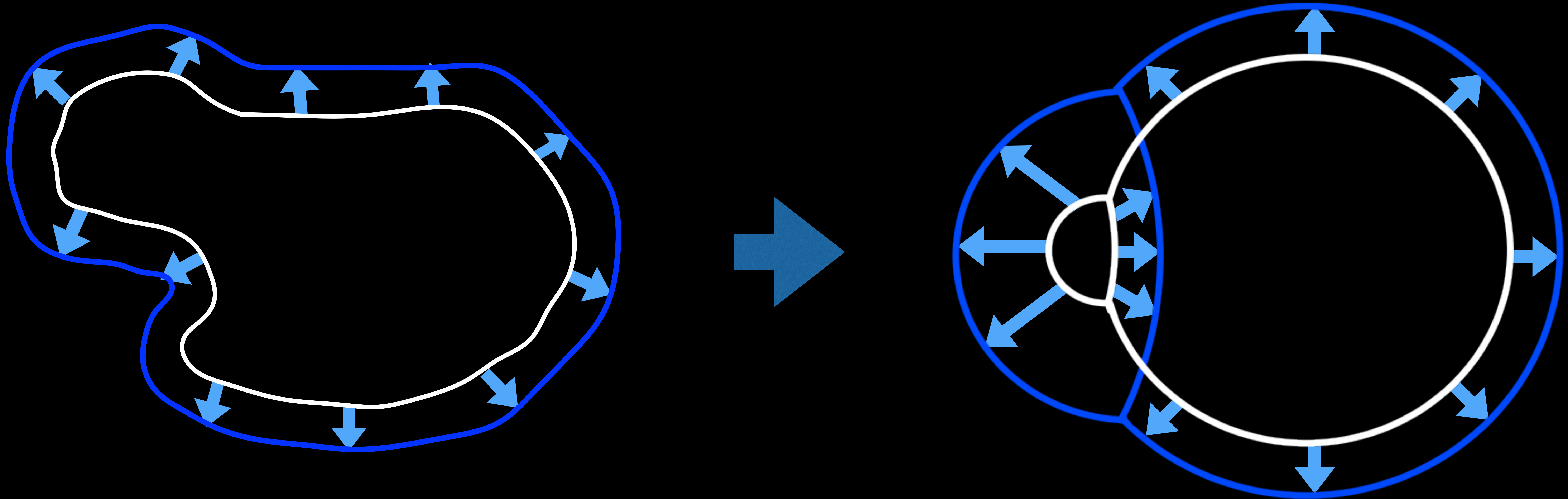
$$\frac{d^2\boldsymbol{x}}{dt^2} = -\beta\frac{\partial A}{\partial \boldsymbol{x}} + \Delta p \boldsymbol{n}$$

Müller's Volume Preservation [2006]



Move each point toward the normal direction.
The correction amount $\Delta d(t)$ is spatially constant.

Extension to Multiple Regions



The correction amount $\Delta d(t, \boldsymbol{x})$ for each point is related to the pressure difference.

Pressure and Volume Preservation

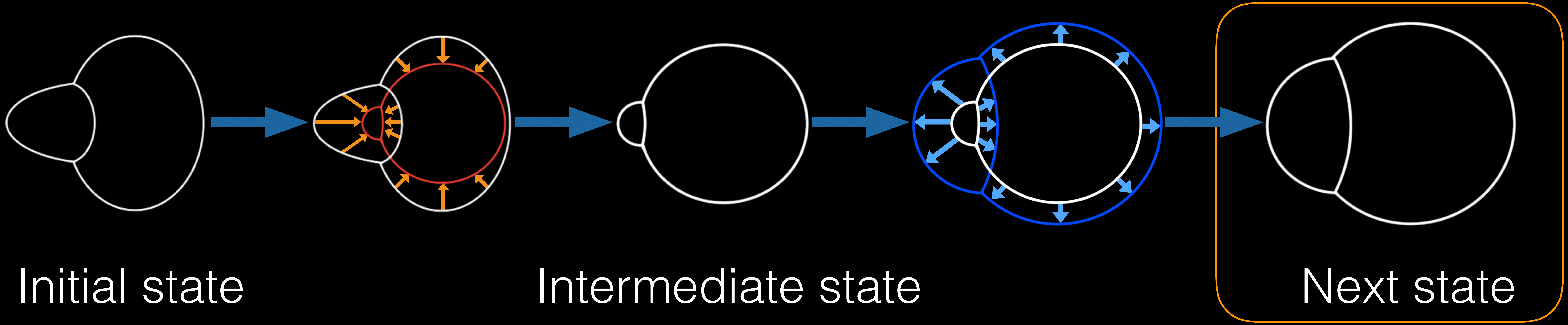
Performing volume preservation

↕

Computing the pressure difference term $\Delta p \boldsymbol{n}$

Assumptions
- incompressibility
- constant pressure per region

After the Second Step



Evolution by $\dfrac{\partial A}{\partial x}$

Volume preservation

Connection to Physics

The Navier-Stokes equations

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{\sigma H \delta(\boldsymbol{x})}{\rho}\boldsymbol{n} + \frac{1}{\rho}\nabla p$$

Our geometric flow

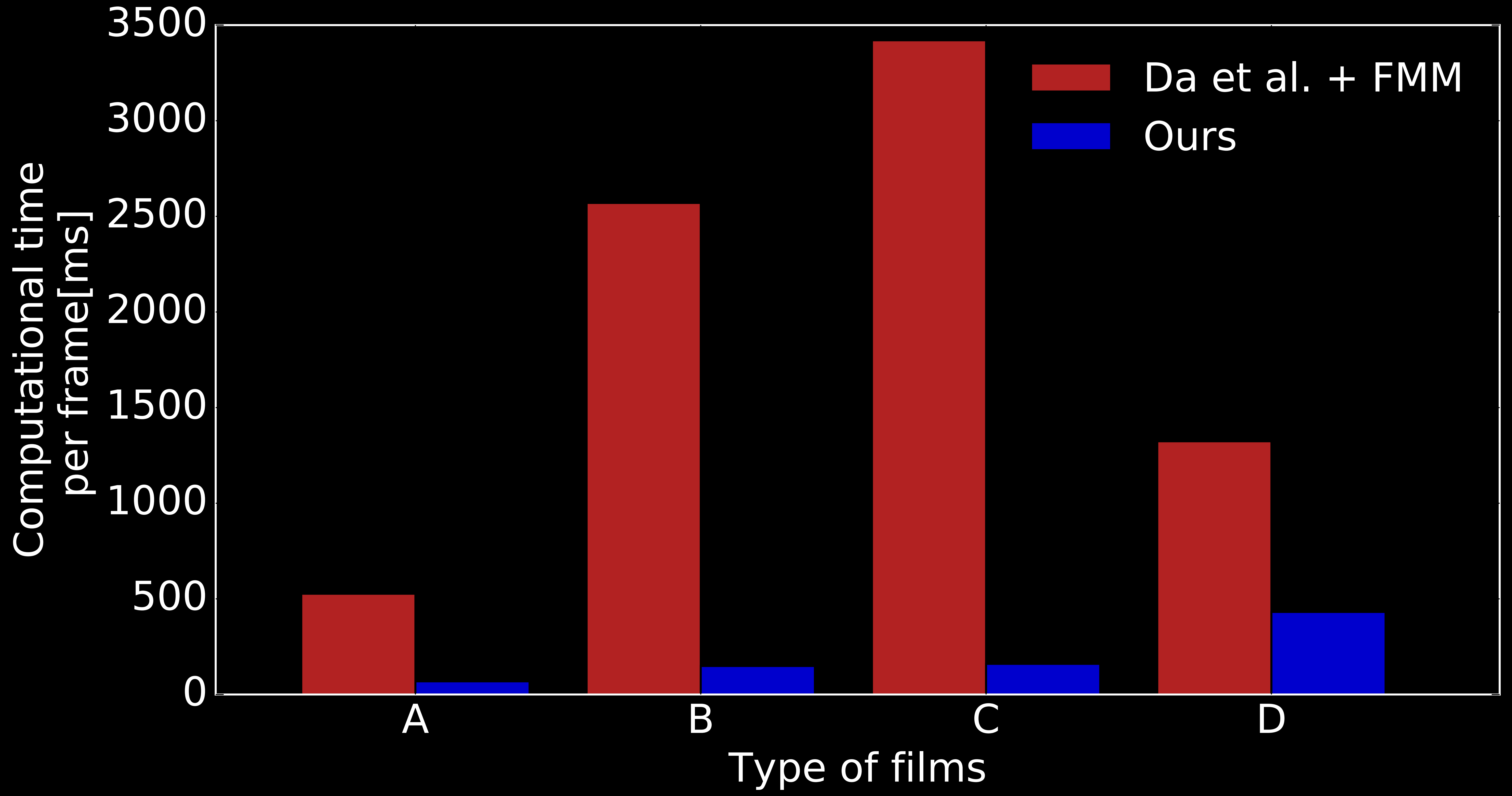
$$\frac{d^2\boldsymbol{x}}{dt^2} = -\beta\frac{\partial \mathcal{A}}{\partial \boldsymbol{x}} + \Delta p \boldsymbol{n}$$

Assumptions

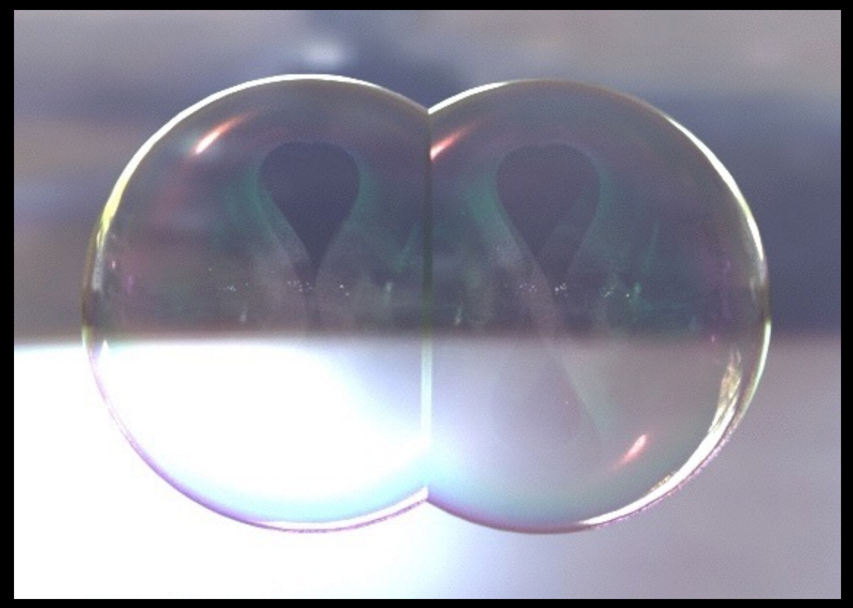
- infinitesimal thickness
- constant pressure per region

Results

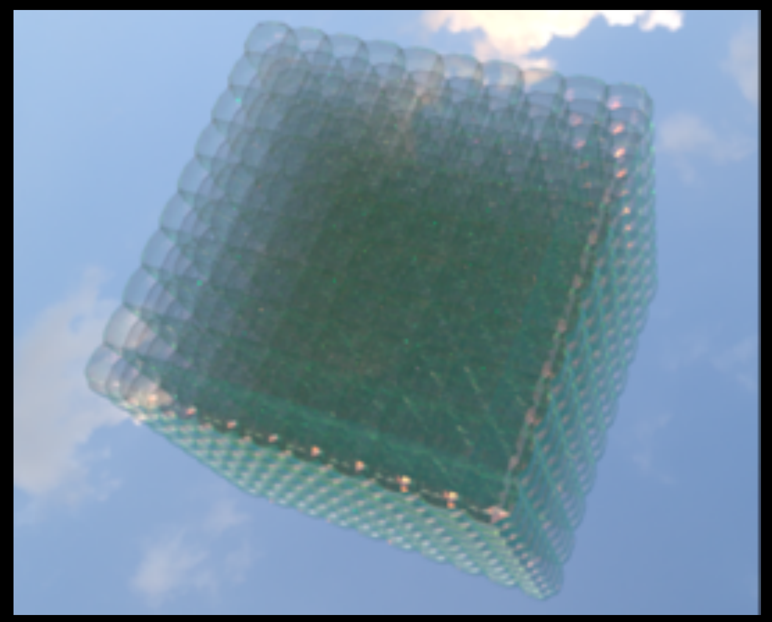
Computational Timings



A



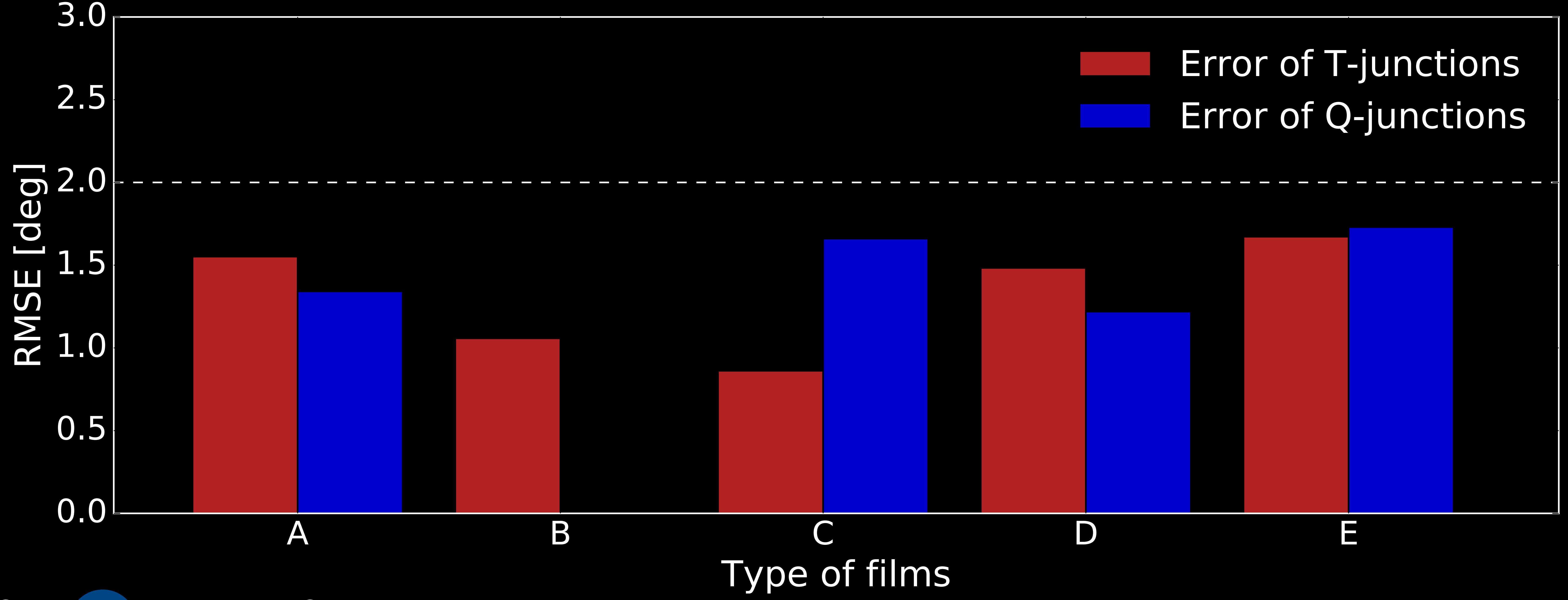
B

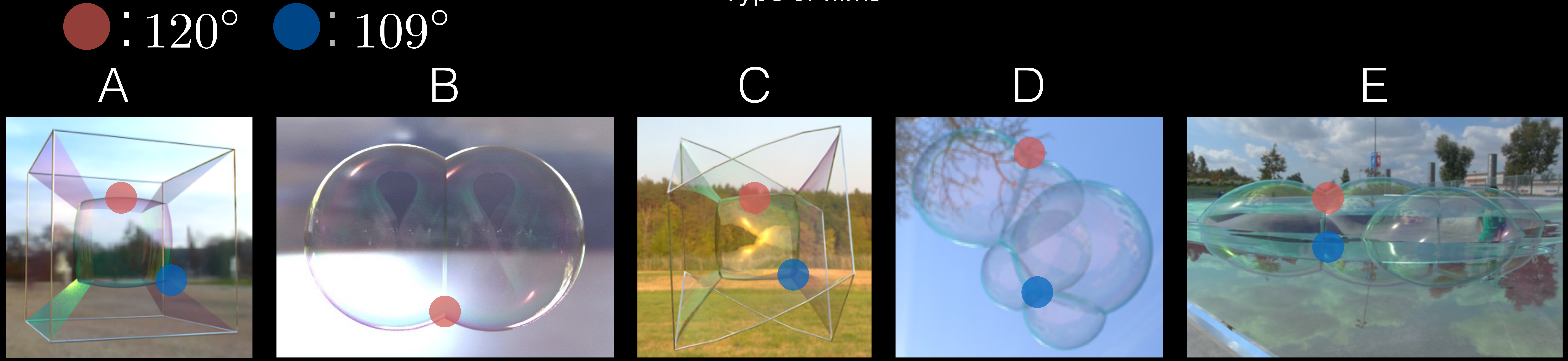


C

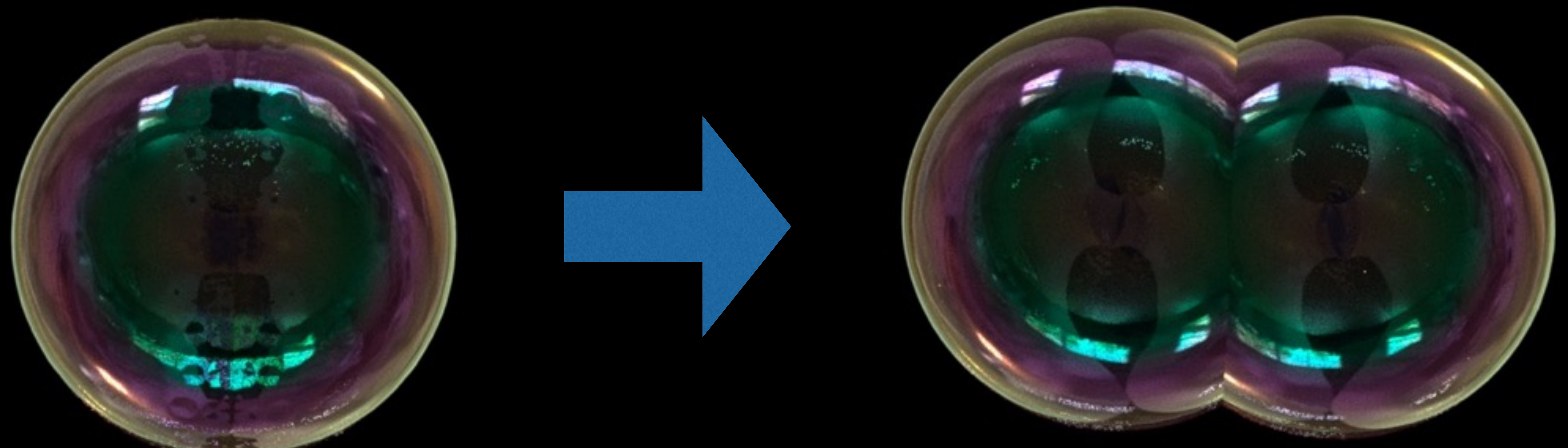
D

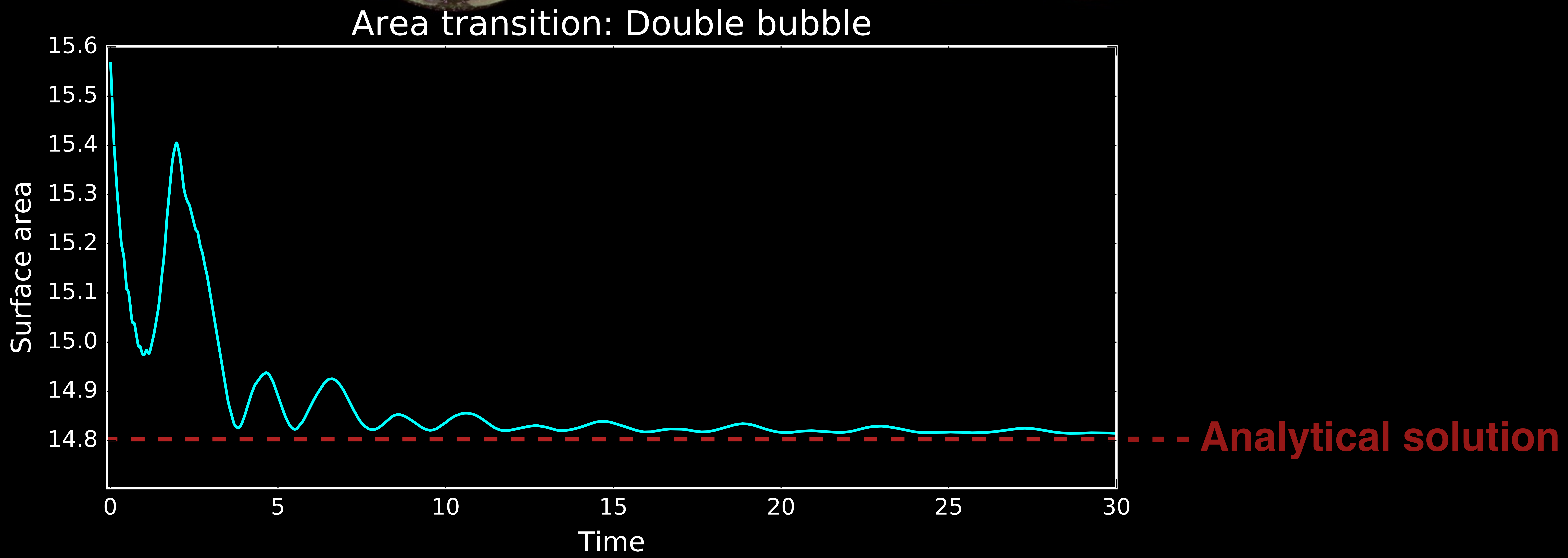
Plateau's Laws





Convergence of Surface Area





Volume Control

Achieved simply by changing the initial inner volumes.

External Force

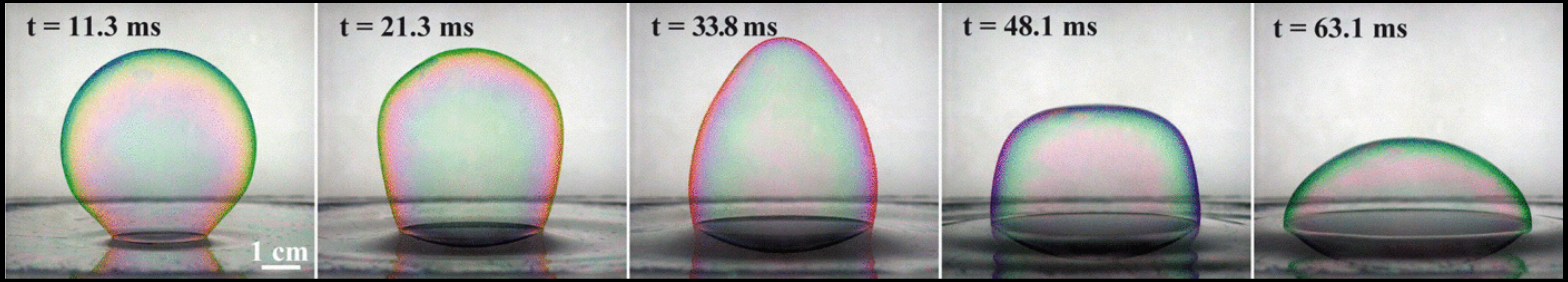
Not well handled in previous work

Da et al. 2015 : Velocity is determined by circulation
→ Cannot directly add external force as acceleration

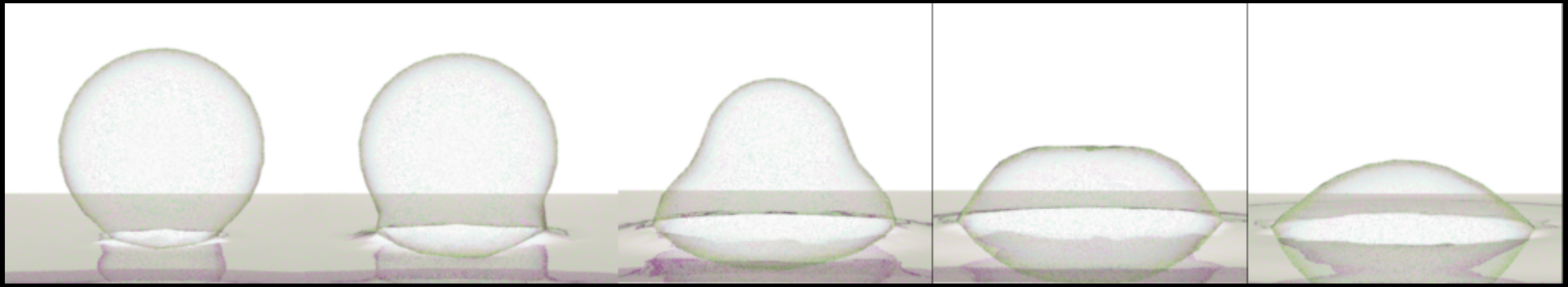
Ours: Velocity is determined by acceleration
→ Straightforward to add external force

External Force

Comparison to Real Footage



experiment [Pucci et al. 2015]



our simulation

Conclusion

Efficient simulation method for soap film dynamics

Mathematical contributions:

- Volume preserving hyperbolic geometric flow for multiple surfaces
- Numerical solver for Plateau's problem, even with presence of external force

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Acknowledgements

- Sigurd Ofstad, for discussion and experiments
- Hisanari Otsu, for customization of Mitsuba renderer
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- Jamorn Sriwasansak, for the advice on the Thai language
- Nikon Corporation, for the support by grants

Source Code is Available

https://github.com/sdsgisd/HGF