# Compressed Sensing: Day 2
## Proof structure

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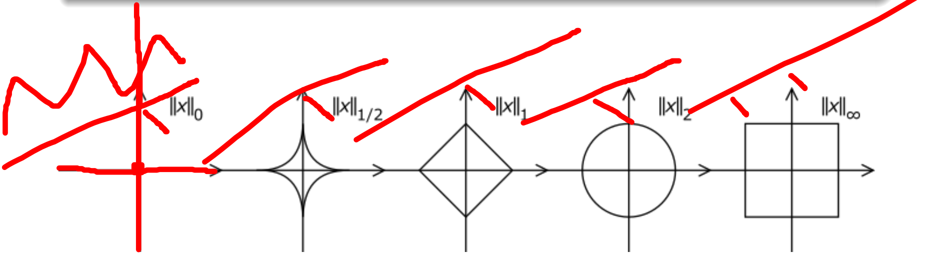
# Norm minimization

**Definition (p-norm minimization)**

$$(P_p)\begin{cases}\min_{g\in\mathbb{C}^N} & \|g\|_p\\ s.t. & \hat{g}=\hat{f} \quad in\ \Omega\,.\end{cases}$$

|\Omega| >= 2 |T|



# When do minimizers coincide?

**Definition ($(P_1)$ minimizer)**

$$f^\sharp \in \operatorname{argmin}(P_1)\,.$$

- When is $f^\sharp = f$?
- What if $\Omega$ is random?

\tau

# Reconstruction with high probability

Fix $f \in \mathbb{C}^N$, $\tau \in (0,1)$ and $M > 0$.

**Theorem ([CRT06, Theorem 1.3])**

*Denote* $T := \operatorname{supp} f$. *If*

|\Omega| >= 2 |T|

$$\tau N \geq |T| \log N \frac{1}{\alpha(M)},$$

*then,* $f = f^\sharp$ *with probability at least* $1 - O(N^{-M})$.

# $|\Omega|$ is "at least" to $\tau N$

Recall that $\Omega$, which defines $(P_1)$, is a random set such that $\mathbb{E}[|\Omega|] = \tau N$. Moreover,

**Lemma (Concentration bound)**

$$\mathbb{P}(|\Omega| > (1 - \varepsilon_M)\tau N) \geq 1 - N^{-M},$$

*where* $\varepsilon_M := \sqrt{\frac{2M \log N}{|\tau N|}}$.

# Proof

## Lemma (Concentration bound)

$$\mathbb{P}(|\Omega| > (1-\varepsilon_M)\tau N) \geq 1 - N^{-M},$$

## Proof.

Since $|\Omega| \sim Bin(N,\tau)$,

$$\begin{aligned}
\mathbb{P}(|\Omega| \leq \tau N - \varepsilon_M \tau N) &\leq \exp\left[\frac{-(\varepsilon_M \tau N)^2}{2\tau N}\right] & ; \textit{Chernoff bound} \\
&= \exp\left[-M \log N\right] & ; \varepsilon_M := \sqrt{\frac{2M\log N}{|\tau N|}} \\
&= N^{-M}.
\end{aligned}$$

□

# Reconstruction with high probability

Fix $f \in \mathbb{C}^N$, $\tau \in (0,1)$ and $M > 0$.

**Theorem ([CRT06, Theorem 1.3])**

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*then,* $f = f^\sharp$ *with probability at least* $1 - O(N^{-M})$.

Fourier data

# Sensing basis

Let $f \in \mathbb{C}^N$. We sense

$$\{\hat{f}(\omega) : \omega \in \Omega\} = \{\langle \mathcal{F}_{N \to \{\omega\}}, f \rangle : \omega \in \Omega\}.$$

- Could we sense something else?
- Can we use other device to sense the signal $f$?
- What if we sense $\{\langle \varphi, f \rangle : \varphi \in \Omega\}$ with $\Omega \subseteq \Phi$ random?

# Description basis

Let $f \in \mathbb{C}^N$. Recall that

**Definition (Support)**

$$\operatorname{supp} f := \{t \in [N] : f(t) \neq 0\}.$$

Again,

$$\{t \in [N] : f(t) \neq 0\} = \{t \in [N] : \langle e_t, f \rangle \neq 0\}.$$

- Could we describe differently?
- Can we use other device to express the signal $f$?
- What if we care about sparsity in $\Psi$: $\{\psi \in \Psi : \langle \psi, f \rangle \neq 0\}$?

# How similar are two basis?

**Definition (Coherence)**

Let $\Phi, \Psi$ be two basis of $\mathbb{C}^N$.

$$\mu(\Phi, \Psi) := \sqrt{N} \max_{\phi \in \Phi, \psi \in \Psi} |\langle \phi, \psi \rangle| \,.$$

The coherence measures the largest correlation between any two elements of $\Phi$ and $\Psi$.

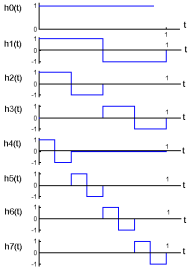
**Exercise (Bounds for coherence)**

$$\mu(\Phi, \Psi) \in [1, \sqrt{N}] \,.$$

# Examples

Possible basis are the following.

- Spikes: $\{e_j = (\delta_j(t))_{t\in[N]} : j \in [N]\}$.
- Fourier: $\left\{\left(\frac{1}{\sqrt{N}}e^{-i\omega_j t}\right)_{t\in[N]} : \omega_j = \frac{2\pi j}{N}, j \in [N]\right\}$.
- Haar, $N = 2^m$: $\{\gamma_{j,k} : 2^j + k \in [N]\}$, e.g.

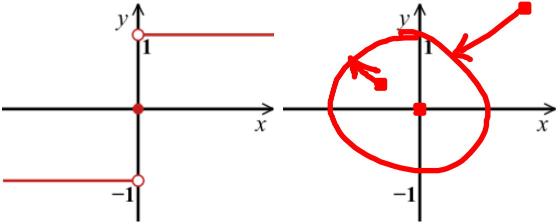


# More notation

**Definition (Sign)**

Let $x \in \mathbb{C}$, define

$$\operatorname{sign} x := \begin{cases} \frac{x}{|x|} & ; x \neq 0 \\ 0 & ; \sim \end{cases} .$$



# Minimizer for given basis

**Definition (1-norm minimization with custom basis)**

$$(P_1(\Psi,\Phi)) \begin{cases} \min_{g\in\mathbb{C}^N} & \sum_{\psi\in\Psi} |\langle \psi, g\rangle| \\ s.t. & \langle \phi, g\rangle = \langle \phi, f\rangle \quad \forall \phi \in \Omega \subseteq \Phi\,. \end{cases}$$

= ||g||_1

\hat{g} = \hat{f}

**Definition**

$$f^\sharp_{\Psi,\Phi} \in \operatorname{argmin}(P_1(\Psi,\Phi))\,.$$

# General result

Random sensing \Omega, fixed signal f

\Phi = \Psi
Coherence = \sqrt{N}

**Theorem ([CR07, Theorem 1.1])**

*Denote* $T := \{\psi : \langle \psi, f \rangle \neq 0\}$. *If*

$$\tau N \geq \mu^2(\Phi, \Psi)|T| \log N \frac{1}{\alpha(M)},$$

*then,* $f = f^\sharp_{\Psi,\Phi}$ *with probability at least* $1 - O(N^{-M})$.

# General result: Deterministic set, random signal

Let $C_0, C_1$ be fixed numerical constants. Fix $\Phi, \Psi$ basis of $\mathbb{C}^N$, $T \subseteq \Psi$, $\Omega \subseteq \Phi$.

**Theorem ([CR07, Theorem 1.1])**

*Fix $\delta > 0$. If*

$$|\Omega| \geq \max\left\{ C_0 \mu^2(\Phi, \Psi)|T| \log\left(\frac{N}{\delta}\right), C_1 \log^2\left(\frac{N}{\delta}\right) \right\},$$

*then, $f = f^\sharp_{\Psi,\Phi}$ with probability at least $1 - \delta$.*

Probability on what? $f$ is random in its sign: choose the sign$\langle \psi, f \rangle$ randomly for each $\psi \in T$. Then, the reconstruction holds independently of the magnitud of $\langle \psi, f \rangle$.

# Reconstruction with high probability

Fix $f \in \mathbb{C}^N$, $\tau \in (0,1)$ and $M > 0$.

**Theorem ([CRT06, Theorem 1.3])**

*Denote* $T := \operatorname{supp} f$. *If*

$$\tau N \geq |T| \log N \frac{1}{\alpha(M)},$$

*then,* $f = f^\sharp$ *with probability at least* $1 - O(N^{-M})$.

# Reduction to $(P_1)$

Let $f \in \mathbb{C}^N$ and $\Omega$ non-empty. Denote $T := \operatorname{supp} f$.

**Lemma (Sufficient conditions)**

*If $\mathcal{F}_{T\to\Omega} : \mathbb{C}^T \to \mathbb{C}^\Omega$ is injective and there exists $P \in \mathbb{C}^N$ such that*

$$\begin{cases} \hat{P}(\omega) = 0 & ; \omega \in \Omega^c \\ P(t) = \operatorname{sign} f(t) & ; t \in T \\ |P(t)| < 1 & ; t \in T^c\,, \end{cases}$$

*then, $f = f^\sharp$ is the unique minimizer of $(P_1)$.*

# Proof: $f$ is a minimizer

**Proof.**

Let $g \in \mathbb{C}^N$ such that $\hat{g}|_\Omega = \hat{f}|_\Omega$. Define $h := g - f$. We will show that $\|g\|_1 \geq \|f\|_1$.

Let $t \in T$. Then,

$$\begin{aligned} |g(t)| &= |f(t) + h(t)| \\ &= ||f(t)| + h(t)\overline{\text{sign } f(t)}| && ; \cdot \overline{\text{sign } f(t)} \\ &\geq |f(t)| + Re(h(t)\overline{\text{sign } f(t)}) && ; |\cdot| \geq Re(\cdot) \\ &= |f(t)| + Re(h(t)\overline{P(t)}) && ; P(t) = \text{sign } f(t). \end{aligned}$$

Therefore, for $t \in T$, $|g(t)| \geq |f(t)| + Re(h(t)\overline{P(t)})$.

$(\ldots)$

□

# Proof: $f$ is a minimizer

**Proof.**

We will show that $\|g\|_1 \geq \|f\|_1$.
Let $t \in T^c$. Then,

$$\begin{aligned} |g(t)| &= |h(t)| \\ &\geq |h(t)\overline{P(t)}| && ; |P(t)| < 1 \\ &\geq Re(h(t)\overline{P(t)}) && ; |\cdot| \geq Re(\cdot)\,. \end{aligned}$$

Therefore, for $t \in T^c$, $|g(t)| \geq |f(t)| + Re(h(t)\overline{P(t)})$.
$(\ldots)$

□

# Proof: $f$ is a minimizer

**Proof.**

Then,

$$\|g\|_1 \geq \|f\|_1 + \sum_{t \in [N]} Re(h(t)\overline{P(t)}) \,.$$

Note that $\hat{h}|_\Omega = \hat{g}|_\Omega - \hat{f}|_\Omega = 0$. Moreover, by Parseval's formula,

$$\sum_{t \in [N]} Re(h(t)\overline{P(t)}) = \frac{1}{N} \sum_{\omega} Re(\hat{h}(\omega)\overline{\hat{P}(\omega)}) = 0 \,,$$

\Omega

\Omega^c

since either $\hat{h}$ or $\hat{P}$ vanishes for each $\omega$.
Thus,

$$\|g\|_1 \geq \|f\|_1 \,.$$

□

# Proof: $f$ is the unique minimizer

**Proof.**

Assume $\|g\|_1 = \|f\|_1$. By the previous inequalities that $|h(t)| = |h(t)\overline{P(t)}|$ and so, for $t \in T^C$,

$$h(t) = 0\,.$$

Since $\hat{h}|_\Omega = 0$ and $\mathcal{F}_{T\to\Omega}$ is injective (and linear),

$$h = g - f \equiv 0\,.$$

□

# Reduction to $(P_1)$

Let $f \in \mathbb{C}^N$ and $\Omega$ non-empty. Denote $T := \operatorname{supp} f$.

**Lemma (Sufficient conditions)**

*If $\mathcal{F}_{T\to\Omega} : \mathbb{C}^T \to \mathbb{C}^\Omega$ is injective and there exists $P \in \mathbb{C}^N$ such that*

$$\begin{cases} \hat{P}(\omega) = 0 & ; \omega \in \Omega^c \\ P(t) = \operatorname{sign} f(t) & ; t \in T \\ |P(t)| < 1 & ; t \in T^c, \end{cases}$$

*then, $f = f^\sharp$ is the unique minimizer of $(P_1)$.*

# Clever construction

**Definition (Extrapolation)**

Denote the extrapolation by zero by the operator $\iota\colon \mathbb{C}^T \to \mathbb{C}^N$.

Its adjoint $\iota^*\colon \mathbb{C}^N \to \mathbb{C}^T$ is the restriction map. Note that $\iota^*\iota\colon \mathbb{C}^T \to \mathbb{C}^T$ is the identity map.
We will prove that there is a suitable operator $H\colon \mathbb{C}^T \to \mathbb{C}^N$ such that

$$P := \left(\iota - \frac{1}{|\Omega|}H\right)\Big(\underbrace{\iota^*\iota}_{\mathrm{Id}} - \frac{1}{|\Omega|}\iota^*H\Big)^{-1}\iota^* \operatorname{sign} f$$

works for our purposes.

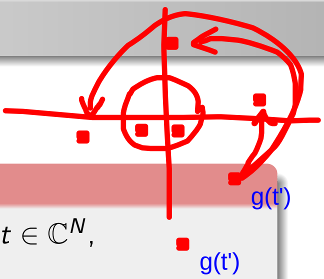
# Why can we make it work?

Recalling the sufficient conditions, we would have to prove that

$$\begin{cases} \hat{P}(\omega) = 0 & ; \omega \in \Omega^c \\ P(t) = \operatorname{sign} f(t) & ; t \in T \\ |P(t)| < 1 & ; t \in T^c \,, \end{cases}$$

(and that $\mathcal{F}_{T \to \Omega}$ is injective) occurs with high probability.
We will prove that if $\left(\iota^* \iota - \frac{1}{|\Omega|} \iota^* H\right)$ is invertible, then $\mathcal{F}_{T \to \Omega}$ is injective, so we can concentrate on $P$.

# Definition




## Definition (White noise operator)

Define the operator $H\colon \mathbb{C}^T \to \mathbb{C}^N$ by, for all $t \in \mathbb{C}^N$,

$$Hg(t) := -\sum_{\omega\in\Omega}\sum_{\substack{t'\in T\\ t\neq t}} e^{i\omega(t-t')}g(t')\,.$$

## Exercise

*Prove that $\iota^* H\colon \mathbb{C}^T \to \mathbb{C}^T$ is self-adjoint.*

# What will we prove?

**Lemma (Constrained Fourier transform)** \hat{P} is supported in \Omega

*For all $g \in \mathbb{C}^T$, the Fourier transform of $\left(\iota - \frac{1}{|\Omega|} H\right) g$ (which is an element of $\mathbb{C}^N$) is supported in $\Omega$.*

**Lemma (sign on $T$)**

*For all $t \in T$, we have that $P(t) = \operatorname{sign} f(t)$.*

**Lemma (Small magnitude)**

*With high probability $|P(t)| < 1$, for all $t \in T^c$.*

**Lemma (Well defined and injectivity)**

*The operator $\left(\iota^* \iota - \frac{1}{|\Omega|} \iota^* H\right)$ is invertible with high probability, and its invertibility implies that $\mathcal{F}_{T \to \Omega}$ is injective.*

# Easy proof I

**Lemma (Constrained Fourier transform)**

*For all $g \in \mathbb{C}^T$, the Fourier transform of $\left(\iota - \frac{1}{|\Omega|} H\right) g$ (which is an element of $\mathbb{C}^N$) is supported in $\Omega$.*

**Constrained Fourier transform.**

Fix $g \in \mathbb{C}^T$. Note that, for all $t \in [N]$,

$$\left(\iota - \frac{1}{|\Omega|} H\right) g(t) = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} \sum_{t' \in T} e^{i\omega(t-t')} g(t') = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} e^{i\omega t} \hat{g}(\omega).$$

Then, the Fourier transform of $\left(\iota - \frac{1}{|\Omega|} H\right) g$ is supported in $\Omega$. □

# Easy proof II

**Lemma (sign on $T$)**

*For all $t \in T$, we have that $P(t) = \operatorname{sign} f(t)$.*

**sign on $T$.**

Fix $t \in T$. Note that,

$$\begin{aligned} P(t) &= \iota^* P(t) && ; t \in T \\ &= \iota^* \left(\iota - \frac{1}{|\Omega|} H\right) \left(\iota^* \iota - \frac{1}{|\Omega|} \iota^* H\right)^{-1} \iota^* \operatorname{sign} f(t) && ; \textit{def } P \\ &= \iota^* \operatorname{sign} f(t) \\ &= \operatorname{sign} f(t) && ; t \in T. \end{aligned}$$

□

# More complex proof

**Lemma (Small magnitude)**

*With high probability* $|P(t)| < 1$, *for all* $t \in T^c$.

**Proof.**

Next week. □

# Easy proof III

## Lemma (Well defined and injectivity)

*The operator* $\left(\iota^*\iota - \frac{1}{|\Omega|}\iota^* H\right)$ *is invertible with high probability, and its invertibility implies that* $\mathcal{F}_{T\to\Omega}$ *is injective.*

## Proof.

Invertibility with high probability will be proven next week. For now, let us prove the implication. Since
$\left(\iota - \frac{1}{|\Omega|}H\right) g(t) = \frac{1}{|\Omega|}\sum_{\omega\in\Omega} e^{i\omega t}\hat{g}(\omega)$, we have that

$$\iota^*\iota - \frac{1}{|\Omega|}\iota^* H = \frac{1}{|\Omega|}(\mathcal{F}_{T\to\Omega})^*\mathcal{F}_{T\to\Omega}\,.$$

Then, the invertibility of $\iota^*\iota - \frac{1}{|\Omega|}\iota^* H$ implies that $\mathcal{F}_{T\to\Omega}$ is injective. □

# Summary

**Lemma (Small magnitude (next week))**

*For all $t \in T^c$, we have that $|P(t)| < 1$ occurs with high probability.*

**Lemma (Invertibility (next week))**

*The operator $\iota^* \iota - \frac{1}{|\Omega|} \iota^* H$ is invertible with high probability.*

# What will we prove?

**Lemma (Constrained Fourier transform)**

*For all $g \in \mathbb{C}^T$, the Fourier transform of $\left(\iota - \frac{1}{|\Omega|} H\right) g$ (which is an element of $\mathbb{C}^N$) is supported in $\Omega$.*

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