Compressed Sensing: Day 4
Complete proof and more

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Properties of this week

Lemma (Small magnitude)

For all $t \in T^c$, we have that $|P(t)| < 1$ occurs with high probability.

Lemma (Invertibility)

The operator $\iota^*\iota - \frac{1}{|\Omega|}\iota^* H$ is invertible with high probability.

Sparsity

Recall that an assumption in the main theorem is $\tau N \geq |T| \log N \frac{1}{\alpha(M)}$, which can be written as

E |\Omega|

$$|T| \leq \alpha(M) \frac{\tau N}{\log N}.$$

"f" is sparse enough

Last lecture, we had to ask that

0.42 = \alpha

$$|T| \leq \frac{(1-\varepsilon_M)^2 \alpha^2}{\gamma^2} \frac{\tau N}{n}.$$

n = M \log N

Notation

$$n := M \log N \, .$$

$$\varepsilon_M := \sqrt{\frac{2M \log N}{\tau N}} = \sqrt{\frac{2n}{\tau N}} \, . \qquad < 1$$

$$\alpha := 0.42 \, .$$

$$\tau \leq 0.44 \, .$$

Previous results

Under the previous assumptions,

$$\mathbb{P}(|\Omega| \geq (1-\varepsilon_M)\tau N) \in 1 - O(N^{-M})\,.$$

$$\mathbb{P}(\|\iota^* H\|_F \leq (1-\varepsilon_M)\alpha|\Omega|) \in 1 - O(N^{-M})\,.$$

0.42

Properties of this week

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For all $t \in T^c$, we have that $|P(t)| < 1$ occurs with high probability.

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The operator $\iota^*\iota - \frac{1}{|\Omega|}\iota^* H$ is invertible with high probability.

Tuesday's recap

Define

$$R_n := \sum_{m=1}^{\infty} \left(\frac{1}{|\Omega|^n} (\iota^* H)^n \right)^m .$$

If $\alpha \in (0,1)$ and $\|\iota^* H\|_F \leq \alpha |\Omega|$, then

$$\|R_n\|_\infty \leq \sqrt{|T|} \frac{\alpha^n}{1 - \alpha^n} .$$

Also,

$$\left(\iota^* \iota - \frac{1}{|\Omega|} \iota^* H \right)^{-1} = (\iota^* \iota + R_n) \sum_{m=0}^{n-1} \left(\frac{1}{|\Omega|} \iota^* H \right)^m .$$

Separation

We can write

$$P = \left(\iota - \frac{1}{|\Omega|}H\right)\left(\iota^*\iota - \frac{1}{|\Omega|}\iota^* H\right)^{-1} \iota^* \operatorname{sign} f$$

$$= \left(\iota - \frac{1}{|\Omega|}H\right)(\iota^*\iota + R_n)\sum_{m=0}^{n-1}\left(\frac{1}{|\Omega|}\iota^* H\right)^m \iota^* \operatorname{sign} f \, .$$

So that, for $t \in T^c$,

$$P(t) = -\frac{1}{|\Omega|}H\sum_{m=0}^{n-1}\left(\frac{1}{|\Omega|}\iota^* H\right)^m \iota^* \operatorname{sign} f(t)$$

$$-\frac{1}{|\Omega|}HR_n\sum_{m=0}^{n-1}\left(\frac{1}{|\Omega|}\iota^* H\right)^m \iota^* \operatorname{sign} f(t)$$

$$= P_{n,0}(t) + P_{n,1}(t)\, .$$

Big small

Objective

Definition

$$P_{n,0}(t) := -\frac{1}{|\Omega|} H \sum_{m=0}^{n-1} \left(\frac{1}{|\Omega|} \iota^* H \right)^m \iota^* \operatorname{sign} f(t)$$

$$P_{n,1}(t) := -\frac{1}{|\Omega|} H R_n \sum_{m=0}^{n-1} \left(\frac{1}{|\Omega|} \iota^* H \right)^m \iota^* \operatorname{sign} f(t)$$

Then, for $t \in T^c$,

$$P(t) = P_{n,0}(t) + P_{n,1}(t) .$$

Lemma (Small magnitude)

$$\mathbb{P} \left(\sup_{t \in T^c} |P(t)| \geq 1 \right) \in O(N^{-M}) .$$

The big and the small

Lemma (Separation in big and small)

For all n, $a \in (0,1)$,

big small

$$\mathbb{P}\left(\sup_{t\in T^c} |P(t)| \geq 1\right) \leq \mathbb{P}\left(\|P_{n,0}\|_\infty \geq a\right) + \mathbb{P}\left(\|P_{n,1}\|_\infty \geq 1-a\right)$$

Proof.

P(t) = P_{n, 0}(t) + P_{n, 1}(t)

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in T^c} |P(t)| \geq 1\right) &\leq \mathbb{P}\left(\|P_{n,0}\|_\infty + \|P_{n,1}\|_\infty \geq 1\right)\\ &\leq \mathbb{P}\left(\|P_{n,0}\|_\infty \geq a \text{ or } \|P_{n,1}\|_\infty \geq 1-a\right)\\ &\leq \mathbb{P}\left(\|P_{n,0}\|_\infty \geq a\right) + \mathbb{P}\left(\|P_{n,1}\|_\infty \geq 1-a\right).\end{aligned}$$

□

The big and the small, in general

Consider X, Y positive random variables.

Lemma (Separation in big and small)

For all $a \in (0,1)$,

$$\mathbb{P}(X + Y \geq 1) \leq \mathbb{P}(X \geq a) + \mathbb{P}(Y \geq 1 - a)$$

Proof.

$$\begin{aligned} \mathbb{P}(X + Y \geq 1) &\leq \mathbb{P}(X \geq a \text{ or } Y \geq 1 - a) \\ &\leq \mathbb{P}(X \geq a) + \mathbb{P}(Y \geq 1 - a) \,. \end{aligned}$$

□

The big and the small, example

Consider $X, Y \sim Exp(1)$. Then,

Gamma(2, 1) 0.606 0.606

$$\mathbb{P}\,(X + Y \geq 1) \approx 0.735 \leq 1.212 \approx \mathbb{P}(X \geq 1/2) + \mathbb{P}(Y \geq 1/2)\,.$$

In the other hand, if $X = Y$, then

$$\mathbb{P}\,(X + Y \geq 1) \approx 0.606 \leq 1.212 \approx \mathbb{P}(X \geq 1/2) + \mathbb{P}(Y \geq 1/2)\,.$$

2Y

The big and the small, example II

Consider $X, Y \sim \begin{cases} 1/2 & w.p. 1/2 \\ 0 & w.p. 1/2\,. \end{cases}$.

Then

0.5 0.5

$$\mathbb{P}(X+Y \geq 1) = 0.25 \leq 1.00 \approx \mathbb{P}(X \geq 1/2) + \mathbb{P}(Y \geq 1/2)\,.$$

In the other hand, if $X = Y$, then

$$\mathbb{P}(X+Y \geq 1) \approx 0.50 \leq 1.00 \approx \mathbb{P}(X \geq 1/2) + \mathbb{P}(Y \geq 1/2)\,.$$

2Y

The big and the small

Recall that, for all n, $a \in (0,1)$,

Lemma (Separation in big and small)

$$\mathbb{P}\left(\sup_{t\in T^c} |P(t)| \geq 1\right) \leq \mathbb{P}\left(\|P_{n,0}\|_\infty \geq a\right) + \mathbb{P}\left(\|P_{n,1}\|_\infty \geq 1-a\right)$$

By choosing $a = 0.91$, we have to prove the following.

Lemma (Bounding the big)

$$\mathbb{P}\left(\|P_{n,0}\|_\infty \geq 0.91\right) \in O(N^{-M}).$$

Lemma (Bounding the small)

$$\mathbb{P}\left(\|P_{n,1}\|_\infty \geq 0.09\right) \in O(N^{-M}).$$

Partitioning the sum

Recall that

$$P_{n,0}(t) := -\frac{1}{|\Omega|} H \sum_{m=0}^{n-1} \left(\frac{1}{|\Omega|} \iota^* H \right)^m \iota^* \operatorname{sign} f(t)$$
$$= - \sum_{m=1}^{n} \left(\frac{1}{|\Omega|} H \iota^* \right)^m \operatorname{sign} f(t) \, .$$

Since the sum of many (random) vectors might be too complex, we will look at each term separately.

Definition

$$Y_m := \left| \left(\frac{1}{|\Omega|} H \iota^* \right)^m \operatorname{sign} f(t) \right| \, .$$

Partitioning 0.91

Fix $t \in T^c$.
Consider $(b_m)_{m \in [1\,..\,n]}$ such that $b_i > 0$ and $\sum_m b_m = 0.91$. Then,

$$\mathbb{P}\left(|P_{n,0}(t)| \leq \sum_{m=1}^{n} Y_m \geq 0.91\right) \leq \sum_{m=1}^{n} \mathbb{P}(Y_m \geq b_m)\,.$$

If we had bounds on the moments of Y_m, we could use Markov inequalities to get an overall bound.

Markov inequality

Fix $t \in T^c$. Assume $n = 2^J - 1$. = M \log N

Lemma ($P_{n,0}$ can be bounded)

If $\mathbb{E}[Y_m^{2K_j}] \leq c_n$ *for* $K_j = 2^{J-j}$ *and* $m \in [2^j \mathinner{..} (2^{j+1} - 1)]$, *then there exists* $a_0 \in (0,1)$ *such that*

$$\mathbb{P}\left(|P_{n,0}(t)| \leq \sum_{m=1}^{n} Y_m \geq 0.91\right) \leq n a_0^{2n} c_n \, .$$

Proof

b_m -> a_j

Proof.

b_m

Choose $(a_j)_{j\in[1\,..\,(J-1)]}$ such that $a_j^{-K_j} = a_0^{-n}$ and $\sum_{j=0}^{J-1} 2^j a_j \leq 0.91$. Notice that

b_m

$$\mathbb{P}\left(\sum_{m=1}^{n} Y_m \geq 0.91\right) \leq \sum_{j=0}^{J-1}\sum_{m=2^j}^{2^{j+1}-1} \mathbb{P}\left(Y_m \geq a_j\right) \quad ; \textit{union bound}$$

2K_j 2K_j

$$\leq \sum_{j=0}^{J-1}\sum_{m=2^j}^{2^{j+1}-1} a_j^{-2K_j}\mathbb{E}\left[Y_m^{2K_j}\right] \quad ; \textit{Markov ineq.}$$

$$\leq \sum_{j=0}^{J-1}\sum_{m=2^j}^{2^{j+1}-1} a_0^{-2n} c_n \quad ; \textit{assumption}$$

$$\leq (\ldots)$$

Proof

Proof.

Choose $(a_j)_{j\in[1\,..\,(J-1)]}$ such that $a_j^{-K_j} = a_0^{-n}$ and $\sum_{j=0}^{J-1} 2^j a_j \leq 0.91$. Notice that

$$\mathbb{P}\left(\sum_{m=1}^{n} Y_m \geq 0.91\right) \leq \sum_{j=0}^{J-1} \sum_{m=2^j}^{2^{j+1}-1} a_0^{-2n} c_n \qquad ; \textit{assumption}$$
$$\leq n a_0^{-2n} c_n \,.$$

By choosing $a_0 = 0.42$, we indeed have that $\sum_{j=0}^{J-1} 2^j a_j = \sum_{j=0}^{J-1} 2^j a_0^{n/K_j} \leq 0.91$. □

Markov inequality

Fix $t \in T^c$. Assume $n = 2^J - 1$.

Lemma ($P_{n,0}$ can be bounded)

If $\mathbb{E}[Y_m^{2K_j}] \leq c_n$ *for* $K_j = 2^{J-j}$ *and* $m \in [2^j \,..\, (2^{j+1}-1)]$, *then there exists* $a_0 \in (0,1)$ *such that*

a_0 = 0.42

$$\mathbb{P}\left(|P_{n,0}(t)| \leq \sum_{m=1}^{n} Y_m \geq 0.91\right) \leq n a_0^{2n} c_n \,.$$

Bound on moments

We will take as given the following result.

Lemma

For $m \in [1 \,..\, n]$, denote j for the integer such that $m \in [2^j \,..\, (2^{j+1} - 1)]$ and define $K_j := 2^{J-j}$, where $n < 2^J < 2n$. Then,

n= M \log N

$$\mathbb{E}[Y_m^{2K_j}] \leq c_n \in O(e^{-n}) .$$

= O(N^{-M})

Proof.

See [CRT06, Appendix, page 36]. □

Overall bound

Notice that the bound is $O(e^{-n})$.
Since $n = M \log N$, $O(e^{-n}) = O(N^{-M})$ and we have that, for each $t \in T^c$,

$$\mathbb{P}(|P_{n,0}(t)| \geq 0.91) \leq n a_0^{2n} c_n \in O(N^{-M})$$

and, since $a_0 = 0.42$ (and therefore $a_0^2 < e^{-1}$),

$$\mathbb{P}(\|P_{n,0}\|_\infty \geq 0.91) \leq n \frac{N}{e^n} c_n \in O(N^{-M}),$$

for $M \geq 1$.

N / N^{M}

n integer = ceil(M \log N)

Discovering the small

Note that, for $t \in T^c$,

small

$$P_{n,1}(t) := \frac{1}{|\Omega|} H R_n \left[\sum_{m=0}^{n-1} \left(\frac{1}{|\Omega|} \iota^* H \right)^m \right] \iota^* \operatorname{sign} f(t)$$

$$= \frac{1}{|\Omega|} H R_n \left(\iota^* \operatorname{sign} f(t) + \sum_{m=1}^{n-1} \left(\frac{1}{|\Omega|} \iota^* H \right)^m \iota^* \operatorname{sign} f(t) \right)$$

$$= \frac{1}{|\Omega|} H R_n \iota^* \left(\operatorname{sign} f(t) + \sum_{m=1}^{n-1} \left(\frac{1}{|\Omega|} H \iota^* \right)^m \operatorname{sign} f(t) \right)$$

$$= \frac{1}{|\Omega|} H R_n \iota^* \left(\operatorname{sign} f(t) + P_{n-1,0}(t) \right) .$$

Big

Therefore,

|| i^* H ||_F

$$\|P_{n,1}\|_\infty \leq \frac{1}{|\Omega|} \|H\|_\infty \|R_n\|_\infty \left(1 + \|P_{n-1,0}\|_\infty \right) .$$

Recalling the bounds

0.42

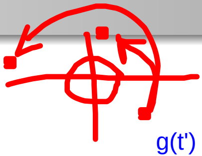
Recall that

- $\mathbb{P}(\|\iota^* H\|_F < (1-\varepsilon_M)\alpha|\Omega|) > 1 - O(N^{-M})$.
- If $\|\iota^* H\|_F < (1-\varepsilon_M)\alpha|\Omega|$, then $\|R_n\|_\infty \leq \sqrt{|T|}\frac{\alpha^n}{1-\alpha^n}$.
- $\mathbb{P}(\|P_{n-1,0}\|_\infty \leq 0.91) > 1 - O(N^{-M})$.

Then, with high probability,

$$\begin{aligned} \|P_{n,1}\|_\infty &\leq \frac{1}{|\Omega|}\|H\|_\infty \|R_n\|_\infty (1 + \|P_{n-1,0}\|_\infty) \\ &\leq \frac{1}{|\Omega|}\|H\|_\infty \sqrt{|T|}\frac{\alpha^n}{1-\alpha^n} 2 . \end{aligned}$$

Magnitud of white noise



Recall that, for $g \in \mathbb{C}^T$,

$$Hg(t) := -\sum_{\omega \in \Omega} \sum_{\substack{t' \in T \\ t \neq t}} e^{i\omega(t-t')} g(t') \, .$$

Therefore, if $\|g\|_\infty \leq 1$,

$$|Hg(t)| \leq \sum_{\omega \in \Omega} \sum_{\substack{t' \in T \\ t \neq t}} |e^{i\omega(t-t')} g(t')|$$

$$\leq \sum_{\omega \in \Omega} \sum_{\substack{t' \in T \\ t \neq t}} 1 \leq |\Omega| |T| \, ,$$

so that

$$\frac{1}{|\Omega|} \|H\|_\infty \leq |T| \, .$$

Overall bound

Then, with high probability,

$$\|P_{n,1}\|_\infty \leq \frac{1}{|\Omega|}\|H\|_\infty \sqrt{|T|}\frac{\alpha^n}{1-\alpha^n}2$$
$$\leq |T|^{3/2}\frac{\alpha^n}{1-\alpha^n}2\,.$$

Recall that $n = M\log N$. Then, if $N^{3/2}\frac{\alpha^n}{1-\alpha^n}2 < 0.09$, we would have that

$$\mathbb{P}(\|P_{n,1}\|_\infty \geq 0.09) \in O(N^{-M})\,.$$

small

Relationship of N and M

Consider the condition

$$N^{3/2}\frac{\alpha^n}{1-\alpha^n}2 = N^{3/2}\frac{\alpha^{M\log N}}{1-\alpha^{M\log N}}2 < 0.09$$

Recalling that $\alpha = 0.42$, we will ask for $M > \log_\alpha(e^{-3/2})$, so that

$$\lim_{N\to\infty} N^{3/2}\alpha^{M\log N} = 0\,.$$

Notice that

$$\begin{aligned}\alpha^n &= e^{n\log\alpha} \\ &\leq e^{-0.87n} && ;\log(0.42) \approx -0.87 \\ &= N^{-0.87M} && ;n = M\log N\,.\end{aligned}$$

Then, we will ask for $-0.87M < -3/2$, i.e.

$$M > 1.724\,.$$

M = 2
--> N > 17

Overall bound

If $N^{3/2}\frac{\alpha^n}{1-\alpha^n}2 < 0.09$, then

$$\mathbb{P}(\|P_{n,0}\|_\infty \geq 0.91), \mathbb{P}(\|P_{n,1}\|_\infty \geq 0.09) \in O(N^{-M}).$$

Therefore,

$$\mathbb{P}\left(\sup_{t\in T^c} |P(t)| \geq 1\right) \in O(N^{-M}).$$

And so we have proven that

Lemma (Small magnitude)

For all $t \in T^c$, we have that $|P(t)| < 1$ occurs with high probability.

Reduction to (P_1)

Let $f \in \mathbb{C}^N$ and Ω non-empty. Denote $T := \operatorname{supp} f$.

Lemma (Sufficient conditions)

If $\mathcal{F}_{T\to\Omega} : \mathbb{C}^T \to \mathbb{C}^\Omega$ is injective and there exists $P \in \mathbb{C}^N$ such that

$$\begin{cases} \hat{P}(\omega) = 0 & ; \omega \in \Omega^c \\ P(t) = \operatorname{sign} f(t) & ; t \in T \\ |P(t)| < 1 & ; t \in T^c, \end{cases}$$

then, $f = f^\sharp$ is the unique minimizer of (P_1).

Clever construction

Definition (Extrapolation)

Denote the extrapolation by zero by the operator $\iota\colon \mathbb{C}^T \to \mathbb{C}^N$.

Its adjoint $\iota^*\colon \mathbb{C}^N \to \mathbb{C}^T$ is the restriction map. Note that $\iota^*\iota\colon \mathbb{C}^T \to \mathbb{C}^T$ is the identity map.
We will prove that there is a suitable operator $H\colon \mathbb{C}^T \to \mathbb{C}^N$ such that

$$P := \left(\iota - \frac{1}{|\Omega|}H\right)\left(\iota^*\iota - \frac{1}{|\Omega|}\iota^* H\right)^{-1}\iota^* \operatorname{sign} f$$

works for our purposes.

Reduction to (P_1)

Let $f \in \mathbb{C}^N$ and Ω non-empty. Denote $T := \operatorname{supp} f$.

Lemma (Sufficient conditions)

If $\mathcal{F}_{T \to \Omega} : \mathbb{C}^T \to \mathbb{C}^\Omega$ is injective and there exists $P \in \mathbb{C}^N$ such that

$$\begin{cases} \hat{P}(\omega) = 0 & ; \omega \in \Omega^c \\ P(t) = \operatorname{sign} f(t) & ; t \in T \\ |P(t)| < 1 & ; t \in T^c , \end{cases}$$

then, $f = f^\sharp$ is the unique minimizer of (P_1).

Reconstruction with high probability

1.724

Fix $f \in \mathbb{C}^N$, $\tau \in (0,1)$ and $M > 0$.

Theorem ([CRT06, Theorem 1.3])

Denote $T := \operatorname{supp} f$. *If*

$$\tau N \geq |T| \log N \frac{1}{\alpha(M)},$$

then, $f = f^\sharp$ *with probability at least* $1 - O(N^{-M})$.

Where

$$\alpha(M) := \frac{(1-\varepsilon_M)^2 \alpha^2}{\gamma^2 M}$$

Sensing and description basis

Theorem ([CR07, Theorem 1.1])

Denote $T := \{\psi : \langle \psi, f \rangle \neq 0\}$. *If*

$$\tau N \geq \mu^2(\Phi, \Psi)|T| \log N \frac{1}{\alpha(M)},$$

then, $f = f^{\sharp}_{\Psi,\Phi}$ *with probability at least* $1 - O(N^{-M})$.

Deterministic sensing and random signal

Let C_0, C_1 be fixed numerical constants. Fix Φ, Ψ basis of $\mathbb{C}^N$, $T \subseteq \Psi$ and $\Omega \subseteq \Phi$.

Theorem ([CR07, Theorem 1.1])

Fix $\delta > 0$. If

$$|\Omega| \geq \max\left\{ C_0 \mu^2(\Phi, \Psi)|T| \log\left(\frac{N}{\delta}\right), C_1 \log^2\left(\frac{N}{\delta}\right) \right\},$$

then, $f = f^{\sharp}_{\Psi,\Phi}$ with probability at least $1 - \delta$.

Probability on what? f is random in its sign: choose the sign$\langle \psi, f \rangle$ randomly for each $\psi \in T$. Then, the reconstruction holds independently of the magnitud of $\langle \psi, f \rangle$.

Perturbed samples

Fix $f \in \mathbb{C}^N$ and Ψ, Φ description and sensing basis.
Consider having access to, for all $\phi \in \Omega$,

$$y_\phi := \langle f, \phi \rangle + \xi_\phi \,,$$

where $\xi_\phi \sim \mathcal{N}(0, \varepsilon^2)$.
Consider

$$(P_1(\Psi,\Phi)) \begin{cases} \min_{g \in \mathbb{C}^N} & \sum_{\psi \in \Psi} |\langle \psi, g \rangle| \\ s.t. & \langle \phi, g \rangle = y_\phi \quad \forall \phi \in \Omega \subseteq \Phi \,. \end{cases}$$

What is the relationship between f and $f^\sharp_{\Psi,\Phi}$?

Relevant support

With a general signal, we can rethink the support and define the following.

Definition (Truncated signal)

Denote

$$f = \sum_{i=1}^{N} \langle \psi_i, f \rangle \psi_i \,,$$

where $(|\langle \psi_i, f \rangle|)_{i \in [1\,..\,N]}$ is decreasing. Then, for $T \in [1\,..\,N]$, define

$$f_T := \sum_{i=1}^{T} \langle \psi_i, f \rangle \psi_i \,.$$

Continuity in reconstruction

The general result states that there is a constant $C_0 > 0$ such that

$$\|f - f^{\sharp}_{\Psi,\Phi}\|_2 \leq \frac{C_0}{\sqrt{|T|}}\|f - f_T\|_1$$

and

$$\|f - f^{\sharp}_{\Psi,\Phi}\|_1 \leq C_0\|f_T - f^{\sharp}_{\Psi,\Phi}\|_1$$

with "high" probability, where "high" depends on $\tau \in (0,1)$ and $T \in [1 \,..\, N]$.

Correct minimization problem

How moral is to solve $(P_1(\Psi, \Phi))$? Should we not solve

$$(\tilde{P}_1(\Psi, \Phi)) \begin{cases} \min_{g \in \mathbb{C}^N} & \sum_{\psi \in \Psi} |\langle \psi, g \rangle| \\ s.t. & \|\langle \phi, g \rangle - y_\phi\|_2 \leq \sqrt{N}\varepsilon \quad \forall \phi \in \Omega \subseteq \Phi \,. \end{cases}$$

In this case, on can show that

$$\|f - \tilde{f}_{\Psi,\Phi}\|_2 \leq \frac{C_1}{\sqrt{|T|}} \|f - f_T\|_1 + C_2 \sqrt{N}\varepsilon \,,$$

for some constants $C_1, C_2 > 0$.

Reconstruction with high probability

Fix $f \in \mathbb{C}^N$, $\tau \in (0,1)$ and $M > 0$.

Theorem ([CRT06, Theorem 1.3])

Denote $T := \operatorname{supp} f$. *If*

$$\tau N \geq |T| \log N \frac{1}{\alpha(M)},$$

then, $f = f^\sharp$ *with probability at least* $1 - O(N^{-M})$.

Where

$$\alpha(M) := \frac{(1-\varepsilon_M)^2 \alpha^2}{\gamma^2 M}$$

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