Compressed Sensing Exercise Sheet 1

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1 Exercise Sheet 1

Exercise 1.1. Prove Parseval's formula, which is also called Plancherel's theorem, given by the following statement. For $f, g \in \mathbb{C}^N$,

$$\sum_t f(t)\bar{g}(t) = \frac{1}{N}\sum_\omega \hat{f}(\omega)\bar{\hat{g}}(\omega)$$

where $\hat{f} = \mathcal{F}f$ in the lecture.

Can we say the same for measurements other than $\mathcal{F}$? (what properties of $\mathcal{F}$ did you use in the proof?)

Exercise 1.2. Recall the following theorem presented in the lectures.

Theorem 1.1 (Candes, Romberg and Tao)**.** *Let N be a prime interger and $f \in \mathbb{C}^N$ be a signal, and T be the support of f. If we have measurements $\hat{f}$ on $\Omega \subset \{0, \dots, N-1\}$ with*

$$|T| \leq \frac{1}{2}|\Omega|,$$

then f is exactly recovered from $\hat{f}|_\Omega$.

Give a counter example for non-prime N, i.e., find a triple (T, Ω, N) s.t. $|T| \leq |\Omega|/2$, and $\mathcal{F}_{T\to\Omega}$ is not injective.

Exercise 1.3. This exercise is about a nature of all signals. The *uncertainty principles*, which is an analogy of the one in quantum mechanics, claims that it is impossible to localize a signal both in time and frequency domains at the same time.

Theorem 1.2 (Uncertainty principles)**.** *Let $f \in \mathbb{C}^N$ be a non-zero signal. Then the both of the following hold,*

$$|\mathrm{supp}(f)| + |\mathrm{supp}(\hat{f})| \geq 2\sqrt{N},$$
$$|\mathrm{supp}(f)||\mathrm{supp}(\hat{f})| \geq N.$$

You will immediately see, if you try to minimize the support of a signal in time domain, you will have a large support in frequency domain, and vice-versa. e.g. A Dirac delta gives $|\mathrm{supp}(f)| = 1$ and $|\mathrm{supp}(\hat{f})| = N$.

Define the signal called Dirac's comb by the following. Let N be a perfect square number i.e. $\sqrt{N} \in \mathbb{N}$, and a signal which consists of spikes of unit height and with uniform spacing equal to $\sqrt{N}$, i.e. at times $t = 0, \sqrt{N}, 2\sqrt{N}, \dots, (N - \sqrt{N})$.

(a) Verify that Dirac's comb attains both inequalities tightly.

(b) Dirac's comb f can "slip out" of the recovery in our probabilistic result in the lecture (Theorem 1.6 in the slides), this is one of the reasons why the statement is probabilistic and not deterministic. Find measurements Ω such that $|\Omega|$ is considerably larger than $\sqrt{N} = |\mathrm{supp}(f)|$, e.g. $|\Omega| > 5\sqrt{N}$ for large enough N, but f cannot be reconstructed from $\hat{f}|_{\Omega}$.

Exercise 1.4. This exercise is about reducing a ℓ_1-optimization to an easier problem. Linear programming is an optimization problem that takes the form

$$\begin{cases} (\min_{x\in\mathbb{R}^N} c\cdot x) \text{ or } (\max_{x\in\mathbb{R}^N} c\cdot x) \\ \text{subject to } (A_1x = b_1) \text{ and/or } (A_2x \geq b_2) \text{ and/or } (A_3x \leq b_3) \\ \text{subject to } (x \geq 0) \text{ or } (x \leq 0)\,. \end{cases}$$

(a) Show that our ℓ_1-optimization problem can be written as a linear programming.

(b) Write the exact problem (write down the constrains) for $N = 5, T = \{1, 2\}$ and $\Omega = \{w = i\frac{2\pi}{N} : i \in \{0, 3, 4\}\}$.

Exercise 1.5. Recall the definition of coherence μ and the operator ι and H from the lectures.

(a) Prove that $\iota^*H \colon \mathbb{C}^T \to \mathbb{C}^T$ is self-adjoint.

(b) Prove that

$$\mu(\Phi, \Psi) \in [1, \sqrt{N}]\,.$$