Compressed Sensing Exercise Sheet 2

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2 Exercise Sheet 2

Exercise 2.1. Show the following:

- $\|A+B\|_F \leq \|A\|_F + \|B\|_F$ for $n \times m$ complex valued matrices A and B.
- $\|AB\|_F \leq \|A\|_F \|B\|_F$ for $n \times m$ and $m \times p$ complex valued matrices A and B.

Exercise 2.2. Interestingly, the optimality of a signal for the ℓ_1-problem,

$$\min_{g \in \mathbb{C}^N} \|g\|_1 := \sum_t |g(t)|, \quad \hat{g}|_\Omega = \hat{f}|_\Omega. \tag{2.1}$$

depends only on its support, not its values.

Show that a signal $f \in \mathbb{C}^N$ is the unique minimizer of the problem if, for any non-zero $h \in \mathbb{C}^N$ s.t. $\hat{h}|_\Omega = 0$, it holds

$$\sum_{t \in T} |h(t)| < \sum_{t \in T^c} |h(t)| \tag{2.2}$$

where $T := \mathrm{supp}(f)$. This inequality tells that it is impossible to concentrate “half” of the information, in ℓ_1-norm, on a sparse set T.

Exercise 2.3. Any linear program can be converted to an equivelent problem that takes the *standard form*, i.e

$$\begin{cases} \min_{x \in \mathbb{R}^N} c \cdot x \\ \text{subject to } Ax = b \\ \text{subject to } x \geq 0\,. \end{cases}$$

This can be done by the following procedure:

1. Convert $\max_{x \in \mathbb{R}^N} c \cdot x$ to $\min_{x \in \mathbb{R}^N} -c \cdot x$
2. Convert constraints of the form $Ax \geq b$ to $-Ax \leq -b$.
3. Convert constraints of the form $Ax \leq b$ to $Ax + y = b$ by adding *slack variables* $y \geq 0$.
4. For any variable x that is not constrained in the form $x \geq 0$, express $x = y - z$ with $y, z \geq 0$.

We know at this point that, due to the linearity of the objective function, if the solution exists and is unique, it lies on either of the vertices of the polyhedron illustrated in Figure 1. Therefore, we can obtain the solution by computing the objective function on the vertices.

(a) Show Problem (2.1) can be written as

$$\begin{cases} \min_{t \in \mathbb{R}^N} \mathbf{1} \cdot t \\ \text{subject to } \mathcal{F}_{T \to \Omega} g = \hat{f}|_{\Omega} \\ \text{subject to } g - t \leq 0, g + t \geq 0. \end{cases}$$

You can assume the signal is real-valued.

(b) Convert this into standard form.

(c) If the resulting linear problem has n variables and m constrains, how many vertices does the polyhedron of feasible points have? (you can give an upper bound)

Exercise 2.4. Code compressed sensing via ℓ_1-optimization:

1. Make a random sparse signal.
2. Perform random measurements i.e. randomly decide Ω and compute $\hat{f}|_{\Omega}$.
3. Perform ℓ_1-optimization from Ω and $\hat{f}|_{\Omega}$ (you can solve directly using a convex optimization library, or convert it into a linear programming and solve it using a linear programming library).
4. Compare the reconstructed signal (the solution of the optimization problem) with the original signal f.
5. (Optional) Repeat this steps for other random set of frequencies Ω'.
6. Perform ℓ_2-optimization, instead of ℓ_1-optimization, from Ω and $\hat{f}|_{\Omega}$. Recall that you can solve the optimization problem or use the inverse Fourier transform assuming the coefficients outside of Ω are zero.
7. Compare the results of ℓ_1 and ℓ_2 recoveries.
8. (Optional) Do the same using another measurement matrix e.g. A matrix s.t. each entry is generated from a Gaussian distribution and the rows are orthogonalized.
9. (Super optional) Try 2D image reconstruction.

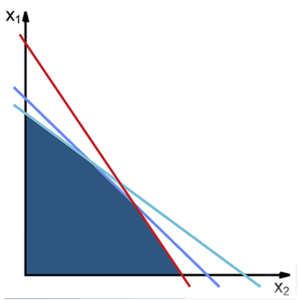

Figure 1: A linear programming in standard form with variables x_1, x_2 and equality constraints represented by the colored lines.