Weak solutions of the semigeostrophic equations from a viewpoint of optimal transport

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This note summarizes my rotation project from March to May 2020 at IST Austria under the advisership of Jan Maas and the support from Lorenzo Portinale. We worked on surveying an application of optimal transport to the semigeostrophic (SG) equations. In this report, we mainly study the papers of Figalli [5] and Benamou and Brenier [2], as well as the works of Cullen [3] and Ambrosio, Colombo, Figalli, and Philippis [1]. The purposes of this report is 1. to provide some backgrounds in the SG equations, and 2. to help the readers obtain better understanding of Figalli's paper by providing detailed discussions and computation. We will skip computation when already given in [5].

In this report, we first introduce the SG system and review a derivation. In the next section, we derive another system called the dual SG system from a view point of optimal transport. Finally, we study construction of weak solutions to the dual SG system.

Assumptions

Throughout this report, we assume the following:

- We deal with only the 2D SG system i.e. there is no gravitational effect or vertical motion.
- We consider the SG system on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.
- Measures are absolutely continuous with respect to the Lebesgue measure unless otherwise stated.

1 The semigeostrophic equations

The semigeostrophic (SG) equations are a set of equations that models the air flow on the earth surface. The SG system is given by,

$$\frac{\partial \nabla p_t}{\partial t} + (u_t \cdot \nabla)\nabla p_t + \nabla^\perp p_t + u_t = 0 \qquad \text{in } [0,\infty) \times \mathbb{R}^2 \tag{1.1}$$

$$\text{div } u_t = 0 \qquad \text{in } [0,\infty) \times \mathbb{R}^2 \tag{1.2}$$

$$p_0 = \bar{p} \qquad \text{given in } \mathbb{R}^2 \tag{1.3}$$

where $u_t : \mathbb{R}^2 \to \mathbb{R}^2$, $p_t : \mathbb{R}^2 \to \mathbb{R}$ are the velocity field and a scalar field called the *pressure*, and $\bar{p}$ is the given initial pressure.

At this point, what these equations describe is not quite clear as it does not have the evolution equation for u_t. In order to understand the mechanism of this system, we start from deriving it from a simpler model called the incompressible Euler equation. This is because we can regard the SG equations as the incompressible Euler equation endowed with the Coriolis effect under some assumptions.

1.1 Derivation

The incompressible Euler equation is a model which describes the dynamics of inviscid and incompressible fluids. It is given by

$$\frac{Du}{Dt} = -\nabla p \tag{1.4}$$

$$\text{div } u = 0 \tag{1.5}$$

where $u : \mathbb{R}^2 \to \mathbb{R}^2$ is the velocity field, and $p : \mathbb{R}^2 \to \mathbb{R}$ physically represents the *pressure* field. The symbol $\frac{D}{Dt} := \partial_t + (\nabla \cdot u)$ is the Lagrange derivative describing the time evolution of a quantity that is stored on each fluid particle advected by the velocity field u. As such, we can interpret that fluid particles are accelerated only by the pressure gradient under the Euler equation.

The next important component is the *Coriolis force*. It is a virtual force caused by the rotation of the earth, which distorts the air flow for an observer on the earth. The Coriolis force acting on a velocity field v is

$$\text{Cor}(v) := c_{\text{cor}} v^\perp = c_{\text{cor}}(v^2, -v^1) \tag{1.6}$$

where c_{cor} is a positive coefficient, which we set 1 for simplicity sake, and $\perp$ denotes 90 degrees rotation. By plugging the Coriolis force to the Euler equation, we have

$$\frac{Du}{Dt} = -\nabla p + u^{\perp}. \tag{1.7}$$

We further introduce the notion of the *geostrophic wind*. The geostrophic wind u_g is a component of the total velocity $u = u_g + u_{ag}$ such that the *geostrophic balance*

$$u_g^{\perp} = \nabla p \tag{1.8}$$

holds. We here employ the *semigeostrophic approximation* [3]: the component u_{ag} called the ageostrophic wind has a small deviation from the geostrophic balance, equivalently u_g is dominant in u, which allows replacing $\frac{Du}{Dt}$ by $\frac{Du_g}{Dt}$ and leads to,

$$\frac{Du_g}{Dt} = -\nabla p + u^{\perp}. \tag{1.9}$$

Note that the advection is performed still along with the total velocity u not u_g i.e. $\frac{Du_g}{Dt} = \partial_t u_g + (\nabla \cdot u)u_g$. Substituting Equation 1.8, we obtain the SG system (Equation 1.1).

From now on, we assume quantities such as p or u are periodic i.e. $p_t(x + n) = p_t(x)$ for $\forall n \in \mathbb{Z}^2$. Namely, we deal with the SG system on the 2D torus $\mathbb{T}^2$. In fact, the periodicity and the boundedness of $\mathbb{T}^2$ play an important role in our construction of solutions.

2 The dual SG

Often times, directly solving a primary problem is challenging, but its dual can be more tractable. In such a case, solutions of the dual problem give a clue for the primary problem. The SG system falls onto this category. Indeed, we can obtain a solution of the primary SG from a solution of the dual SG without further assumptions [5]. In this section, we will derive the dual SG system from a view point of optimal transport.

2.1 Geostrophic coordinates

In the dual SG, two types of coordinate systems play a role. We first regard x as a coordinate system that sticks to particles travelling along with the total velocity u as

$$\frac{Dx}{Dt} = u. \tag{2.1}$$

We here introduce another coordinate system called *the geostrophic coordinates*,

$$y := x + u_g^{\perp}. \tag{2.2}$$

We note that y travels along with the geostrophic wind u_g as we have

$$\frac{Dy}{Dt} = \frac{Dx}{Dt} + \frac{Du_g^{\perp}}{Dt} = u + (-\nabla p + u^{\perp})^{\perp} = u - \nabla^{\perp} p - u = u_g. \tag{2.3}$$

In the sequel, we see that the relation between the two coordinate systems x and y is closely related to optimal transport.

2.2 Modified pressure P

A geophysical criterion called Cullen's stability principle requires that the *geostrophic energy* defined by,

$$E[p_t] := \int \frac{1}{2}(\nabla p_t)^2 dx = \int \frac{1}{2} u_g^2 dx \tag{2.4}$$

should be stable under infinitesimal displacement of particles [4]. Cullen showed that this criterion is satisfied if and only if the *modified pressure*,

$$P_t(x) := p_t(x) + |x|^2/2 \tag{2.5}$$

is convex.

In this section, we consider the rearrangement between particles with the coordinates x and $y = \nabla P_t(x)$. It is expressed by the push-forward measure ρ_t defined by,

$$d\rho_t := (\nabla P_t)_{\#} dx. \tag{2.6}$$

We note that ρ_t is periodic i.e. $\rho_t(A) = \rho_t(A+n)$ for any measurable set A and $n \in \mathbb{Z}^2$ as ρ_t inherits the periodicity of p_t. From this, we observe

$$\int_{[0,1]^2} d\rho_t = \int_{[0,1]^2} dx = 1, \tag{2.7}$$

hence ρ_t and dx are probability measures on $\mathbb{T}^2$.

The famous Brenier theorem connects optimality and convexity of maps for probability measures on $\mathbb{R}^2$. We here use a variant for $\mathbb{T}^2$.

Theorem 2.1 (Ambrosio, Colombo, Figalli, and Philippis. 2011 [1]) *Let μ, ν be probability measure on $\mathbb{T}^2$ and $\mu = f dx$ with some $f > 0$ a.e.. Then there exists a unique [1] convex function $\phi : \mathbb{R}^2 \to \mathbb{R}$ s.t. $\nabla\phi$ is the optimal transport from μ to ν with respect to the quadratic cost, and $\phi - |x|^2/2$ is periodic. Conversely, if there is a convex function ϕ s.t. $\phi - |x|^2/2$ is periodic, then ϕ is optimal transport from μ to ν.*

This theorem allows us to interpret the convexity of P in the context of optimal transport. Recall $y = x + u_g^{\perp}$, then the geostrophic energy reads,

$$\int \frac{1}{2} u_g^2 dx = \int \frac{1}{2}|y - x|^2 dx = \int \frac{1}{2}|\nabla P(x) - x|^2 dx. \tag{2.8}$$

Hence, the convexity of P corresponds to the optimality of the rearrangement between the two coordinates x and y that travel along with u and u_g respectively.

We also note that Theorem 2.1 gives a convex function ϕ with an optimal transport $\nabla\phi$ from $\mu = f dx$ to ν. If $\nu = g dx$ with some g, there exists a unique convex function ψ with the optimal transport $\nabla\psi$ of the opposite direction. As expected, $\nabla\psi = (\nabla\phi)^{-1}$ at the points where both of ψ and ϕ are differentiable. In fact, this holds a.e. since a convex function is differentiable a.e. with respect to the Lebesgue measure. Furthermore, $\psi = \phi^*$ and $\phi = \psi^*$ via the Legendre transform,

$$\phi^*(y) := \sup_x \{x \cdot y - \phi(x)\}. \tag{2.9}$$

[1] up to $\phi + const.$

2.3 Time evolution of $\nabla P_t \# dx$

In the dual SG system, we concern the time evolution of the rearrangement $\rho_t = \nabla P_t \# dx$ in the sense of distribution. From div $u_t = 0$ and $\nabla P_t^* = \nabla P_t^{-1} a.e.$, we can deduce[2] that, for any $\phi \in C_c^\infty(\mathbb{R}^2)$,

$$\frac{d}{dt}\int \phi(y)d\rho_t(y) = \int \nabla\phi \cdot (\nabla P_t^* - y)^\perp d\rho_t(y) := -\int \phi\nabla\cdot((\nabla P_t^* - y)^\perp \rho_t). \tag{2.10}$$

Hence, we have derived the dual SG system,

$$\partial_t \rho_t + \mathrm{div}(U_t\rho_t) = 0 \qquad \text{in } [0,\infty)\times\mathbb{R}^2 \tag{2.11}$$

$$U_t(y) = (\nabla P_t^* - y)^\perp \qquad \text{in } [0,\infty)\times\mathbb{R}^2 \tag{2.12}$$

$$\nabla P_t^* \# \rho_t = dx \qquad \text{in } [0,\infty)\times\mathbb{R}^2 \tag{2.13}$$

$$p_0 = \bar{p} + |x|^2/2 \qquad \text{given in } \mathbb{R}^2 \tag{2.14}$$

Note that $U_t(y) = ((\nabla P_t^* - \mathrm{id})(y))^\perp = (-u_g^2, u_g^1)^\perp = u_g$ is the geostrophic wind. We also note that the dual SG involves incompressibility in its definition as div $U_t = \partial_1\partial_2 P_t^* - \partial_2\partial_1 P_t^* = 0$. Thus Equation 2.11 reduces to $\frac{D\rho_t}{Dt} = 0$, meaning that the density of each fluid particle is constant in time.

As we can see, the dual SG is essentially the transport equation with additional constraints. Therefore, it already looks simpler and more tractable than the primary SG, and we can expect to exploit properties of the transport equation, which has been widely studied.

Finally, we remark that Equation 2.13 is a Monge-Ampère type equation $\rho_t = \det(D^2 P)$ since

$$\int \phi(x)\rho_t(x)dx = \int \phi(\nabla P_t)dx = \int \phi(y)\det(D^2 P)dy, \tag{2.15}$$

or equivalently $\rho_t(\nabla P^*)\det(D^2 P^*) = 1$. As such we can obtain U_t if we could solve this Monge-Ampère equation at each time [9].

3 Transport equation

In the next section, we will study weak solutions of the dual SG. For this purpose, we explore the transport equation, hoping that we can get a solution

[2] For a detailed computation, see [5].

of the dual SG by narrowing down a solution of the transport equation as the dual SG is a special case of the transport equation.

The transport equation is given by

$$\partial_t \sigma_t + \text{div } (v_t \sigma_t) = 0 \tag{3.1}$$

$$\text{div } v_t = 0 \tag{3.2}$$

$$\sigma_0 = \bar{\sigma} \tag{3.3}$$

with the density $\sigma_t : \mathbb{R}^2 \to \mathbb{R}$ and the velocity field $v_t : \mathbb{R}^2 \to \mathbb{R}^2$. By the incompressibility condition, the first equation can be simply written $\frac{D\sigma_t}{Dt} = 0$. We consider a setting in which v_t is given and have some regularity: v_t is Lipschiz in $y \in \mathbb{R}^2$ uniformly with respect to t, meaning the Lipschiz constant is independent of t.

For $\forall y \in \mathbb{R}^2$, we consider a flow $Y : [0, \infty) \times \mathbb{R}^2 :\to \mathbb{R}^2$ given by,

$$\begin{aligned} \dot{Y}(t, y) &= v_t(Y(t, y)) \\ Y(0, y) &= y. \end{aligned} \tag{3.4}$$

We can regard this as an ODE describing the trajectory $Y_y(t) := Y(t, y)$. Then, the classical Cauchy-Lipschitz theorem guaratees the existence of a unique solution $Y_y(t)$ for $t \in [0, \infty)$.

Proposition 3.1 (Cauchy-Lipschitz) *Let $f : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ a function which satisfies the Lipschitz condition uniformly in t. Here T admits ∞. Then the system*

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)) \\ x(0) &= x_0 \end{aligned}$$

has a unique solution $x : [0, T] \to \mathbb{R}^n$.

We now claim that $\sigma_t := Y(t, \cdot)\#\bar{\sigma}$ is the unique solution of the transport equation in the sense of distribution. We can first check for any smooth function ϕ, it holds,[3]

$$\frac{d}{dt} \int \phi \; d\sigma_t + \int \phi \nabla \cdot (v_t \sigma_t) = 0. \tag{3.5}$$

[3]For a detailed computation, see [5].

For the uniqueness, let σ_t be an arbitrary solution. Then we have,

$$\begin{aligned}\frac{d}{dt}\int \phi(y)\ dY(t,\cdot)^{-1}\#\sigma_t(y) &= \frac{d}{dt}\int \phi\circ Y(t,\cdot)^{-1}(y)d\sigma_t(y)\\ &= \frac{d}{dt}\int \phi\circ Y(t,\cdot)^{-1}\circ Y(t,y)d\bar{\sigma}(y)\\ &= \frac{d}{dt}\int \phi(y)d\bar{\sigma}(y) = 0. \end{aligned} \tag{3.6}$$

This means that $Y(t,\cdot)^{-1}\#\sigma_t$ is time-constant. Hence,

$$Y(t,\cdot)^{-1}\#\sigma_t = Y(0,\cdot)^{-1}\#\sigma_0 = \bar{\sigma}, \tag{3.7}$$

meaning that $\sigma_t = Y(t,\cdot)\#\bar{\sigma}$ is uniquely determined by $\bar{\sigma}$ and v_t.

We can also check that the bound of the density is time-invariant:

$$\lambda \le \bar{\sigma} \le \Lambda \Rightarrow \lambda \le \sigma_t \le \Lambda \quad \text{for } \lambda \le \Lambda. \tag{3.8}$$

via a direct computation using $\det \nabla Y(t,\cdot) = 1$ which follows from $\operatorname{div} v_t = 0$. [4] An intuitive explanation is that the fluid particles just move around and never get condensed or sparse under a divergence-free velocity field. Therefore, if the bound of $\bar{\sigma}$ is sharp, so is σ_t. Moreover, the time evolution of σ_t is merely continuous rearrangements of $\bar{\sigma}$.

4 Weak solution of the dual SG

In this section, we study weak solutions of the dual SG. Likewise many non-simple PDEs, the SG system requires "construction of solution". We do so by leveraging the solution of the transport equation. We follow the steps given in [5].

1. Construct approximate piece-wise solutions for $t \in [k\epsilon, (k+1)\epsilon], k \in \mathbb{N}$.

2. Take the limit and see it satisfies the necessary conditions.

[4] For a detailed computation, see [5].

4.1 Approximate solutions

We first make the solution of an approximate system in $t \in [0, \epsilon]$. Let us set the initial density $\rho_0 := \nabla P_0 \# dx = (\nabla p_0 + x)\# dx$ with initial bound

$$\lambda \le \rho_0 \le \Lambda. \tag{4.1}$$

As we have seen, we can expect a well constructed ρ_t inherits the same bound.

Since ρ_0 is absolutely continuous, Theorem 2.1 guarantees the existence of a convex function P_0^* such that ∇P_0^* is an optimal transport map from ρ_0 to dx. We then define a velocity field

$$U_0(y) := (\nabla P_0^*(y) - y)^\perp. \tag{4.2}$$

From this, we further define *time-constant* $U_t^{\epsilon,\delta} \in C^\infty(\mathbb{R}^2)$ for $t \in [0, \epsilon]$ by regularizing U_0 using a mollifier $\chi \in C_c^\infty(\mathbb{R}^2)$ with $\text{supp}\chi = B_1$,

$$U_t^{\epsilon,\delta}(y) := U_0 * \chi_\delta(y) \tag{4.3}$$

where $\chi_\delta(y) := \frac{1}{\delta^2}\chi(\frac{y}{\delta})$ is again a mollifier with $\text{supp}\chi_\delta = B_\delta$. We note that $\text{div } U_t^{\epsilon,\delta} = (\text{div } U_0) * \chi_\delta = 0$.

Likewise we did for the transport equation, let us define a flow

$$\dot{Y}^{\epsilon,\delta}(t, y) = U_t^{\epsilon,\delta}(Y^{\epsilon,\delta}(t, y)) \tag{4.4}$$

$$Y^{\epsilon,\delta}(0, y) = y \tag{4.5}$$

and a density

$$\rho_t^{\epsilon,\delta} := Y^{\epsilon,\delta}(t, \cdot)\#\rho_0, \tag{4.6}$$

for $t \in [0, \epsilon]$. As we have seen, the pair $(\rho_t^{\epsilon,\delta}, U_t^{\epsilon,\delta})$ is the unique solution of

$$\partial_t \rho_t^{\epsilon,\delta} + \text{div}(U_t^{\epsilon,\delta}\rho_t^{\epsilon,\delta}) = 0 \text{ in } [0, \epsilon] \times \mathbb{R}^2 \tag{4.7}$$

and $\lambda \le \rho_t^{\epsilon,\delta} \le \Lambda$.

If we define *time-constant* $P_t^{*,\epsilon,\delta} := P_0^*$ for $t \in [0, \epsilon]$, we have the unique solution of the system,

$$\partial_t \rho_t^{\epsilon,\delta} + \text{div}(U_t^{\epsilon,\delta}\rho_t^{\epsilon,\delta}) = 0 \qquad \text{in } [0, \epsilon] \times \mathbb{R}^2 \tag{4.8}$$

$$U_t^{\epsilon,\delta} = (\nabla P_t^{*,\epsilon,\delta} - y)^\perp * \chi_\delta \qquad \text{in } [0, \epsilon] \times \mathbb{R}^2 \tag{4.9}$$

$$(\nabla P_t^{*,\epsilon,\delta})\#\rho_0^{\epsilon,\delta} = dx \qquad \text{for } t \in [0, \epsilon] \tag{4.10}$$

It looks like we can already have a desired solution if we take $\epsilon = \infty$ and $\delta \to 0$ and do not need to concatenate solutions for $[0, \epsilon], [\epsilon, 2\epsilon], \ldots$. The last condition, however, holds only for $\rho_0^{\epsilon,\delta}$, but not for time-varying $\rho_t^{\epsilon,\delta}$. This is why we need iteration and limit taking.[5]

In order to make the subsequent piece-wise solutions, we mimic the above procedure for $t \in [k\epsilon, (k+1)\epsilon]$ for $k \in \mathbb{N}$. We define for $t \in [\epsilon, 2\epsilon]$,

$$U_t^{\epsilon,\delta}(y) := (\nabla P_\epsilon{}^{*,\epsilon,\delta} - y)^\perp * \chi_\delta(y) \tag{4.11}$$

$$\rho_t^{\epsilon,\delta} := Z^{\epsilon,\delta}(t, \cdot)\#\rho_\epsilon{}^{\epsilon,\delta} \tag{4.12}$$

by using $\rho_\epsilon^{\epsilon,\delta}$ and finding $\nabla P_\epsilon^{*,\epsilon,\delta}$ via Theorem 2.1 again, where $Z_t^{\epsilon,\delta}$ is the solution of the system

$$\dot{Z}^{\epsilon,\delta}(t, y) = U_t^{\epsilon,\delta}(Z^{\epsilon,\delta}(t, y)) \tag{4.13}$$

$$Z^{\epsilon,\delta}(\epsilon, y) = Y^{\epsilon,\delta}(\epsilon, y). \tag{4.14}$$

Again, $(\rho_t^{\epsilon,\delta}, U_t^{\epsilon,\delta})$ is the unique solution of

$$\partial_t \rho_t^{\epsilon,\delta} + \operatorname{div}(U_t^{\epsilon,\delta} \rho_t^{\epsilon,\delta}) = 0 \text{ in } [\epsilon, 2\epsilon] \times \mathbb{R}^2 \tag{4.15}$$

and $\lambda \le \rho_t^{\epsilon,\delta} \le \Lambda$. Iterating this procedure and defining

$$P_t^{*,\epsilon,\delta} := P_{k\epsilon}^{*,\epsilon,\delta} \text{ for } t \in [k\epsilon, (k+1)\epsilon), k \in \mathbb{N}, \tag{4.16}$$

we obtain a soution of the system

$$\partial_t \rho_t^{\epsilon,\delta} + \operatorname{div}(U_t^{\epsilon,\delta} \rho_t^{\epsilon,\delta}) = 0 \qquad \text{in } [0, \infty) \times \mathbb{R}^2 \tag{4.17}$$

$$U_t^{\epsilon,\delta} = (\nabla P_t^{*,\epsilon,\delta} - y)^\perp * \chi_\delta \qquad \text{in } [0, \infty) \times \mathbb{R}^2 \tag{4.18}$$

$$(\nabla P_t^{*,\epsilon,\delta})\#\rho_t^{\epsilon,\delta} = dx \qquad \text{for } t \in \{k\epsilon : k \in \mathbb{N}\} \tag{4.19}$$

$$\lambda \le \rho_t^\epsilon \le \Lambda \qquad \text{in } [0, \infty) \times \mathbb{R}^2. \tag{4.20}$$

Similar to the first iteration, the third condition holds only for discrete time points.

[5] If we make $\nabla P_t^{*,\epsilon,\delta}$ time-varying using Theorem 2.1 for every $\rho_t^{\epsilon,\delta}$, the last condition holds, but the second one fails.

4.2 Limit of the approximate solutions

Our final task is to take the limit $\epsilon, \delta \to 0$ and verify it satisfies necessary conditions. For this purpose, we check

$$\rho_t^{\epsilon,\delta} \rightharpoonup^* \rho_t \qquad \text{in } L^\infty_{loc}([0,\infty] \times \mathbb{R}^2) \tag{4.21}$$

$$U_t^{\epsilon,\delta} \rightharpoonup^* U_t \qquad \text{in } L^\infty_{loc}([0,\infty] \times \mathbb{R}^2; \mathbb{R}^2) \tag{4.22}$$

$$\rho_t^{\epsilon,\delta} U_t^{\epsilon,\delta} \rightharpoonup^* \rho_t U_t \qquad \text{in } L^\infty_{loc}([0,\infty] \times \mathbb{R}^2; \mathbb{R}^2). \tag{4.23}$$

4.2.1 Convergences of $\rho_t^{\epsilon,\delta}$ and $U_t^{\epsilon,\delta}$

We start from the weak * convergences of $\rho_t^{\epsilon,\delta}$ and $U_t^{\epsilon,\delta}$ in $L^\infty_{loc}([0,\infty] \times \mathbb{R}^2)$. Ambrosio, Colombo, Figalli, and Philippis [1] showed

$$|\nabla P_t^{\epsilon,\delta} - y| \leq \text{diam } \mathbb{T}^2 = \sqrt{2}/2, \tag{4.24}$$

from which the boundeddness of $U_t^{\epsilon,\delta}$ follows.

Then we use the Banach-Alaoglu theorem which states that any bounded sequence in a Banach space has a convergent subsequence in the weak * topology. We need a trick to apply this theorem since the space L^∞_{loc} is not Banach as the norm $\sup_{K:compact} \| \cdot \|_{L^\infty(K)}$ is not bounded.

We exploit $L^\infty_{loc}([0,\infty] \times \mathbb{R}^2) = \cap_n L^\infty([0,n] \times B_n)$ where $L^\infty([0,n] \times B_n)$ is decreasing for n. Since the sequence $\rho_t^{\epsilon,\delta}$ is in $L^\infty([0,1] \times B_1)$, there must be a convergent subsequence $\rho_t^{\epsilon,\delta,1}$ in $L^\infty([0,1] \times B_1)$. Then $\rho_t^{\epsilon,\delta,1}$ is in $L^\infty([0,2] \times B_2)$, there must be a convergent subsequence $\rho_t^{\epsilon,\delta,2}$. Repeating this, the resulting sequence has a convergent subsequence in any n, which means convergent in $L^\infty_{loc}([0,\infty] \times \mathbb{R}^2)$.

4.2.2 Convergence of $\rho_t^{\epsilon,\delta} U_t^{\epsilon,\delta}$

The most technical part is the weak * convergence of $\rho_t^{\epsilon,\delta} U_t^{\epsilon,\delta}$ in $L_{loc}([0,\infty] \times \mathbb{R}^2)$. To prove this, we claim

$$\rho_t^{\epsilon,\delta} \rightharpoonup^* \rho_t \qquad \text{in } L^p_{loc}([0,\infty), W^{s,q}_{loc}(\mathbb{R}^2)^*), \tag{4.25}$$

$$U_t^{\epsilon,\delta} \rightharpoonup^* U_t \qquad \text{in } L^\infty([0,\infty), W^{s,q}_{loc}(\mathbb{R}^2)). \tag{4.26}$$

If the above two claims are true, we have

$$\rho_t^{\epsilon,\delta} U_t^{\epsilon,\delta} \rightharpoonup^* \rho_t U_t \quad \text{in } L^\infty_{loc}([0,\infty] \times \mathbb{R}^2; \mathbb{R}^2) \tag{4.27}$$

as it holds for any $\phi \in L^1_{loc}((0,\infty)\times\mathbb{R}^2;\mathbb{R}^2)$,

$$\langle \rho_t^{\epsilon,\delta} U_t^{\epsilon,\delta} - \rho_t U_t, \phi\rangle = \int\int \rho_t^{\epsilon,\delta} U_t^{\epsilon,\delta}\cdot\phi - \rho_t U_t\cdot\phi dxdt = \langle \rho_t^{\epsilon,\delta}\phi, U_t^{\epsilon,\delta}\rangle - \langle \rho_t\phi, U_t\rangle \tag{4.28}$$

and putting $f_t^{\epsilon,\delta} := \rho_t^{\epsilon,\delta}\phi$, we obtain

$$|\langle f_t^{\epsilon,\delta}, U_t^{\epsilon,\delta}\rangle - \langle f_t, U_t\rangle| \le |\langle f_t^{\epsilon,\delta} - f_t, U_t^{\epsilon,\delta}\rangle| + |\langle f_t, U_t^{\epsilon,\delta} - U_t\rangle| \to 0. \tag{4.29}$$

Weak * convergence of $U_t^{\epsilon,\delta}$ in $L^\infty([0,\infty), W^{s,q}_{loc}(\mathbb{R}^2))$

For any t, we have $U_t^{\epsilon,\delta} \in W^{1,1}_{loc}(\mathbb{R}^2)$ since $U_t^{\epsilon,\delta}$ is bounded and

$$\int_{B_R} \|D^2 P_t^{*,\epsilon,\delta}\| dy \le \int_{B_R} \Delta P_t^{*,\epsilon,\delta} dy \le \int_{\partial B_R} |\nabla P_t^{*,\epsilon,\delta}| dy \le C_R. \tag{4.30}$$

where C_R is the Lipschitz constant of $\nabla P_t^{*,\epsilon,\delta}$ on B_R. We here used the facts that the Hessian of a convex function is positive semi-definite, and the matrix norm $\|A\| := \sup_{\|x\|=1}\|Ax\|$ of a symmetric positive semi-definite matrix is bounded by the trace. The third inequality is due to the divergence theorem. Finally, the fractional Sobolev embedding theorem [10] gives

$$U_t^{\epsilon,\delta} \in L((0,\infty), W^{1,1}_{loc}(\mathbb{R}^2)) \subset L((0,\infty), W^{s,q}_{loc}(\mathbb{R}^2)) \tag{4.31}$$

for $s \in (0,1)$ and $1 \le q < \frac{2}{1+s}$. In our case, we set $s = \frac{1}{2}, q = \frac{5}{4}$. In fact, the setting $s < 1$ and $q > 1$ is important in the sequel.

Convergence of $\rho_t^{\epsilon,\delta}$ in $L^p_{loc}([0,\infty), W^{s,q}_{loc}(\mathbb{R}^2)^*)$

To get the convergence, we aim to apply the Aubin-Lions theorem.

Theorem 4.1 (Aubin-Lions) *Let $X0 \subset\subset X \subset X1$ be Banach spaces, and $1 \le p, q < \infty$. Then the embedding*

$$\{f \in L^p([0,T];X_0) \mid \dot f \in L^q([0,T];X_1)\} \hookrightarrow L^p([0,T];X)$$

is compact.

For our target $L^p_{loc}([0,\infty), W^{s,q}_{loc}(\mathbb{R}^2)^*)$ namely $X = W^{s,q}_{loc}(\mathbb{R}^2)^*$, we would like to find nice X_0, X_1. First, from $L^\infty([0,\infty), L^\infty(\mathbb{R}^2)) \subset L^p_{loc}([0,\infty), L^p_{loc}(\mathbb{R}^2))$ and the uniform boundedness of ρ^ϵ_t in time, it follows

$$\rho^\epsilon_t \in L^p_{loc}([0,\infty), L^p_{loc}(\mathbb{R}^2)). \tag{4.32}$$

Next, by the uniform boundedness of $\rho^{\epsilon,\delta}_t$ and $U^{\epsilon,\delta}_t$, it holds for any smooth function $\phi : \mathbb{R}^2 \to \mathbb{R}$,

$$\frac{d}{dt}\int \phi\rho^{\epsilon,\delta}_t dy = \int -\text{div}\ (\rho^{\epsilon,\delta}_t U^{\epsilon,\delta}_t)\phi\ dy = \int U^{\epsilon,\delta}_t \cdot \nabla\phi\rho^{\epsilon,\delta}_t dy \le C\|\phi\|_{W^{1,1}(\mathbb{R}^2)}. \tag{4.33}$$

This means $\partial_t\rho^{\epsilon,\delta}_t \in W^{1,1}(\mathbb{R}^2)$ uniformly in t. Combining $W^{1,q}_{loc}(\mathbb{R}^2) \subset W^{1,1}_{loc}(\mathbb{R}^2)$ for $q \ge 1$ and $L^\infty([0,\infty), X) \subset L^p_{loc}([0,\infty), X)$ for any Banach space X, it holds

$$\partial_t\rho^{\epsilon,\delta}_t \in L^\infty([0,\infty), W^{1,1}_{loc}(\mathbb{R}^2)^*) \subset L^p_{loc}([0,\infty), W^{1,q}_{loc}(\mathbb{R}^2)^*). \tag{4.34}$$

Now we have

$$X_0 = L^p_{loc}(\mathbb{R}^2),\ X = W^{s,q}_{loc}(\mathbb{R}^2)^*,\ X_1 = W^{1,q}_{loc}(\mathbb{R}^2)^*. \tag{4.35}$$

It is clear that $X \subset X_1$ for $s \le 1$. However, while $X_0 \subset X$ is readily seen for certain parameters p, q, s, its compact embedding is not quite straightforward. To show this, we use the next theorem [10].

Theorem 4.2 *Let $s \in (0,1), p \in [1,\infty), q \in [1,p]$ and $\Omega \in \mathbb{R}^2$ a bounded extension domain for $W^{s,q}$. Then, for any bounded family $\mathcal{F} \subset W^{s,q}(\Omega)$, the inclusion $\imath(\mathcal{F})$ is precompact in $L^p(\Omega)$*

We cannot simply apply this theorem as L_{loc} or W_{loc} are not Banach spaces. Likewise we did before, a diagonal argument for $W^{s,q}_{loc}(\mathbb{R}^2) = \cap_n W^{s,q}(B_n)$ and $L^p_{loc}(\mathbb{R}^2) = \cap_n L^p(B_n)$ solves it. To do this, we note that open balls are extension domains for $W^{s,q}$ by the next lemma [10].

Lemma 4.1 *Let $s \in (0,1), q \in [1,\infty)$ and $\Omega \subset \mathbb{R}^n$ be an open Lipschitz set with bounded boundary. Then Ω is an extension domain for $W^{s,q}$.*

To show $X^* \subset\subset X_0^*$, we first take an arbitrary bounded sequence f_j in $W^{s,q}_{loc}(\mathbb{R}^2)$, which is in $W^{s,q}(B_n)$ for any n. Applying Theorem 4.2, f_j has a subsequence $f_{j,1}$ s.t. $\imath(f_{j,1})$ converges in $L^{p'}(B_1)$. Then, the sequence $f_{j,1}$ has a subsequence $f_{j,2}$ s.t. $\imath(f_{j,2})$ converges in $L^{p'}(B_2)$. By repeating this, the resulting sequence is convergent in $L^{p'}(B_n)$ for any n, which means in $L^{p'}_{loc}(\mathbb{R}^2)$. We thus verified the desired compact embedding, which allows us to apply the Aubin-Linos theorem to obtain

$$\rho_t^{\epsilon,\delta} \to \rho_t \quad \text{in } L^p_{loc}([0,\infty), W^{s,q}_{loc}(\mathbb{R}^2)^*) \tag{4.36}$$

up to subsequence.

4.2.3 Convergence of $U_t^{\epsilon,\delta}$ to $(\nabla P_t^* - y)^\perp$

Finally, we need to verify the convergence,

$$U_t^{\epsilon,\delta} = (\nabla P_t^{*,\epsilon,\delta} - y)^\perp * \chi_\delta \to (\nabla P_t^* - y)^\perp \quad \text{in } L^1_{loc}(\mathbb{R}^2). \tag{4.37}$$

For this, we check $\nabla P_t^{*,\epsilon,\delta} \to \nabla P_t^*$ in $L^1_{loc}(\mathbb{R}^2)$ by the next theorem.

Theorem 4.3 (Stability of optimal transport [11]) *Let X be a locally compact polish space, $c : X \times X \to \mathbb{R}_{\geq 0}$ be a lower semicontinuous cost function, ν_k be a sequence of probability measures on X converging weakly to ν, and $T_k : X \to X$ be a sequence of optimal transport maps from μ to ν_k. Then it holds,*

$$\forall \epsilon,\ \mu(\{x \in X \ s.t.\ d(T(x), T_k(x) \geq \epsilon)\}) \longrightarrow 0 \qquad \text{as } k \to \infty. \tag{4.38}$$

To apply this theorem, we show $\rho_t^{\epsilon,\delta} \to^* \rho_t$ in $L^\infty(\mathbb{R}^2)$. From (4.36), it follows that, for a.e. t, there exists a convergent subsequence $\rho_t^{\epsilon,\delta} \to \rho_t$ in $W^{s,q}_{loc}(\mathbb{R}^2)^*$. From the existence of the limit and the boundedness of $\rho_t^{\epsilon,\delta}$ in $L^\infty(\mathbb{R}^2)$, the Banach-Alaoglu theorem gives a convergent subsequence in $L^\infty(\mathbb{R}^2)$.

Finally, applying the theorem allows us to make the region on which $|\nabla P_t^{*,\epsilon,\delta} - \nabla P_t^*| > \epsilon'$ as small as we want for any ϵ'. From this fact and the boundedness,

$$|\nabla P_t^{*,\epsilon,\delta}(y) - \nabla P_t^*(y)| \leq |\nabla P_t^{*,\epsilon,\delta}(y) - y| + |\nabla P_t^*(y) - y| \leq \sqrt{2}, \tag{4.39}$$

it follows

$$\nabla P_t^{*,\epsilon,\delta} \to \nabla P_t^* \quad \text{in } L^1_{loc}(\mathbb{R}^2) \text{ a.e. } t \leq 0. \tag{4.40}$$

We thus obtained all the necessary convergences and verified that the limit of the approximate solution is indeed a weak solution of the dual SG system.

5 Weak solution of the primary SG

We finally review a result [5] without a proof. We can actually obtain a solution of the primary SG system from a solution of the dual SG system.

Theorem 5.1 *Let ρ_t and U be a solution of the dual SG system, and let $P_t^* := \mathrm{id} - U^\perp$ and $P_t := (P_t^*)^*$. Then the pair*

$$p_t := P_t(x) - \frac{|x|^2}{2} \tag{5.1}$$

$$u_t := -D^2 P_t^*(\nabla P_t) \cdot \left(\partial_t \nabla P_t + (\nabla P_t - x)^\perp\right) \tag{5.2}$$

is a solution of the primary SG system in the sense of distribution.

6 Future work

We conclude this report by listing a few directions for the future work:

- SG on the sphere (or more generally a compact manifold)

  Figalli [5] studied the SG equations on the torus. Since the earth is more like a sphere, working on a sphere is a natural direction. For the torus, the boundedness $|\nabla P_t^* - y| \leq \mathrm{diam}\ \mathbb{T}^2$ and the periodicity of quantities are essential. Investigating how replacing these conditions by those of the sphere affects the proof would be interesting.

- Relation to Arnold's theorem or the Benamou-Brenier formula

  The relation of these theorems to the Euler equation has been well-studied. Exploring variants of these results for the SG is interesting as the dual SG can be seen as a non-linear version of the Euler equation.

- Consistency with the transport equation

  Each piece-wise solution of the approximate dual SG system inherits its good properties from the solution of the transport equation e.g. the sharp bound of ρ_t is time preserving. However, whether they still remain in the limit of the concatenated solution is unclear. Aside from this, the interpolation maps $s\nabla P_t + (1-s)\mathrm{id}$ should be entropy-preserving in s at each time t as ∇P_t merely performs a rearrangement between two densities that evolve under incompressible fields u and u_g.

We could investigate these properties, and if it is not as expected, we could work toward *better solutions*.

- Numerical schemes via the Monge-Ampère equation.

  Previous work developed 1. finite difference schemes for SG [7, 4] and 2. numerics for the Euler equation and the Monge-Ampère equation via semi-discrete optimal transport [8, 6]. However, the SG with semi-discrete optimal transport has not been well explored, to the best of our knowledge. Another possibility for a numerical scheme of the SG is to simply discretize Figalli's arguments. In that case, the main challenges would be actually finding a map ∇P_t^* and a subsequence of the approximate solutions that converges to the desired limit as only their existences are guaranteed.

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